Fano-Mori Contractions

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Reference:

[A-W1] *On contractions of smooth varieties*,
to appear on Journal of Algebraic Geometry.

[A-W2] *A view on contractions of higher dimensional varieties*,
to appear on the Proceedings of Santa Cruz -1995.

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A *contraction* $\varphi : X \to Z$ is a proper map of normal irreducible varieties with connected fibers.
We study the case X smooth; F denotes a fiber.

The contraction φ is called *Fano-Mori* (or also *extremal*) if the anti-canonical divisor $-K_X$ is φ-ample.

The map can be *birational* or of *fiber type*, i.e. $dimX > dimZ$.
The *exceptional locus* E of a birational contraction φ is the smallset subset of X such that φ is an isomorphism on $X \setminus E$.
If $codimE \geq 2$ then φ is called *small*.

We will assume that φ is *elementary*, i.e. $Pic(X/Z) = \mathbf{Z}$.

Example

–A Fano variety X with the constant map,

–a minimal ruled surface,

–the blow-up (of a smooth subvarieties in a smooth variety).

S. Mori classified these contractions if $dimX = 3$; besides the blow-up and the Fano 3-folds one has:

– $\varphi : X \supset E \to Z \ni z$, $E \simeq \mathbf{P}^2$ and $N_{E/X} = \mathcal{O}(-2)$

– $\varphi : X \supset E \to Z \ni z$, $E \simeq \mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{S}_2$ and $N_{E/X} = \mathcal{O}(-1)$

– conic bundle and del Pezzo fibration.

F-M contractions are the base of the Minimal Model Program.

? $dimX \geq 4$?

Birational case:

Small contraction [Kawamata, Inventiones 89] proved that it is unique and of the type:
-$\varphi : X \supset E \to Z \ni z$ where $E \simeq \mathbf{P}^2$, $N_{E/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Divisorial contractions, i.e $dimE = 3$; let $S = \varphi(E)$ and $s := dimS$.

–If $s = 0$ then E is a del Pezzo variety of dimension 3.

Problem: Is the non normal possible? (Takagi).

-If $s = 1$ then E is either a $\mathbf{P}^2$-bundle or a quadric bundle and φ is (locally) the blow-up of a smooth curve in a hypersurface of $\mathbf{C}^5$. All fibers are reduced ,i.e. there is no double $\mathbf{P}^2$. (this is due to Takagi and A-W).

-If $s = 2$ then generically φ is a smooth blow-up of a smooth surface. <u>We can have isolated two dimensional fiber</u>: this is what we will mainly discuss in the next.

Example (of birational type):

Blow-up a smooth surface in a 4-fold with an ordinary double point; i.e.

$$S := \{x = z = w = 0\} \subset Z := \{xy - zt + w^2\}$$

$$\varphi : X := Bl_S Z \to Z.$$

A direct computation shows that X is smooth and that $\varphi^{-1}((0,0,0,0,0)) = \mathbf{P}^2$.

Example (of fiber type):

Let L_1, L_2, L_3 be three general plane in $\mathbf{P}^3$ and let $\mathbf{P}^2$ be the base of the net $\mathcal{L} = \Sigma t_i L_i$.
Consider the incidence variety

$$X := \{(p, L); p \in L\} \subset \mathbf{P}^3 \times \mathbf{P}^2.$$

Then the projection $\varphi : X \to \mathbf{P}^3$ is a F-M contraction which is a $\mathbf{P}^1$-bundle generically and has a fiber $= \mathbf{P}^2$ over the point intersection of the L_i.

In coordinates:
assume $\mathbf{P}^3 = [z_0, z_1, z_2, z_3], \mathbf{P}^2 = [t_1, t_2, t_3]$, $L_i = z_i$, $i = 1, 2, 3$.
Then $X = \{t_1 z_1 + t_2 z_2 + t_3 z_3 = 0\} \subset \mathbf{P}^2 \times \mathbf{P}^3$.

If we blow-up a smooth surface S in X meeting the general fiber in one point we obtain a smooth conic bundle $Y \to \mathbf{P}^3$ with a two dimensional reducible fiber and with discriminant locus $\Delta = \varphi(S)$. On Y there is another F-M contraction, of birational type; the one of the previous example.

Let $\varphi : X \to Z$ be a F-M contraction, X smooth, $dimX \leq 4$ and F be a two dimensional fiber.

To describe and classify these object we run over a "Program" which is made of the following main steps/results:

(A) Theorem: the description of the fiber.
If φ is birational then F is isomorphic to $\mathbf{P}^2$ or to a quadric (i.e. either $\mathbf{F}_0$ or $\mathbf{S}_2$ or $\mathbf{P}^2 \cup \mathbf{P}^2$).
If φ is a conic fibration a similar theorem holds with more cases, three of which are doubtfull; this last was proved indepedently by Kachi)

(B) Theorem: spannedness (and description) of the conormal bundle.
If φ is birational the conormal bundle of F into X, $N^*_{F/X}$, is spanned by global sections.
If φ is a conic fibration the same is true under some other assumption, for insatnce if F is $\mathbf{P}^2$ or a quadric.

Corollary. *In particular F has the fiber structure which is reduced and with no embedded components.*

(C) Construction of examples and description of the singularities of Z

(D) Theorem: unicity of φ in an analytic (or formal) neighborhood of F.

Two words on the proof of the first two theorem; some of the ingredients

Vanishing theorems and more generally the *Base Point free technique* (Enriques, Kodaira, Bombieri, Kawamata, Viehweg, Kollar,...).
This is in general used to prove the relative base point free of $-K_X$, or of some multiple of it, or the existence of some good divisor in the linear system $-K_X$ (elephant).

Theorem (A-W, Duke J. *93). Let F be a fiber of a Fano-Mori contraction of dimension $> dimX - dimZ$. Assume that $dimF \leq 2$.
Then $-K_X$ is φ spanned.*

Remark-Problem: If $dimF \geq 3$ this is false.
One can then hope to get elephants;
this can be done in the case $F = X$ up to $dimX = 4$
(Fujita, Wilson and Mella).

Existence and deformation of rational curves in the fibers.
With this we can eliminate many case.

Rank two vector bundle theory.
We use theorems of Van de Ven, Forster-Hirschowitz-Schneider, Maruyama.

Example
Let $\mathcal{E}$ be a vector bundle over a smooth variety F and let $\mathbf{V}(\mathcal{E}) := Spec(S(\mathcal{E}))$ be the *total space of the dual* $\mathcal{E}^*$.

If $S^k(\mathcal{E})$ is *generated by global sections* for some $k > 0$ let

$$\varphi : \mathbf{V}(\mathcal{E}) \to Z = Spec(\bigoplus_{k\geq 0} H^0(F, S^k(\mathcal{E})),$$

be the map associated to the evaluation of $S^k(\mathcal{E})$. It is a contraction, the collapsing of the zero section , $F_0 := F$, of the total space $\mathbf{V}(\mathcal{E})$ to the vertex z of a cone Z.
The normal bundle of F_O into $\mathbf{V}(\mathcal{E})$ is $\mathcal{E}^*$.

- If $-K_Y - det\mathcal{E}$ is ample then φ is a Fano-Mori contraction. The map is birational if the top Segre class of $\mathcal{E}$ is positive (if $rank\mathcal{E} = 2$ then $c_1^2 - c_2 > 0$).

- $\mathbf{P}(\mathcal{O} \oplus \mathcal{E}) := Proj(S(\mathcal{E} \oplus \mathcal{O}_Y))$ is its projective closure. The map φ is the restriction of the map given by the tautological bundle ξ on $Proj(S(\mathcal{E} \oplus \mathcal{O}_Y))$; φ is birational if ξ is big.

- (Grauert criterium). $\mathcal{E}$ is ample if and only if the map φ is an isomorphism outside Y_O.

- The fiber Y_0 of the map φ has the fiber structure (i.e. $\mathcal{I}_{Y_0} = \varphi^{(-1)} m_z \mathcal{O}_X$) if and only if $\mathcal{E}$ is spanned by global section.

Fano Mori Contractions of a 4-fold with a fiber $F = \mathbf{P}^2$.

By the above results $\mathcal{E} = N^*_{F/X}$ is spanned by global sections and, by adjunction formula, $0 \leq c_1(\mathcal{E}) \leq 2$.

Proposition. *Let $\mathcal{E}$ be a rank-2 vector bundle over $\mathbf{P}^2$. Suppose that $\mathcal{E}$ is spanned by global sections and $0 \leq c_1(\mathcal{E}) \leq 2$. Then $\mathcal{E}$ is isomorphic to one of the following bundles.*

n.	c_1	c_2	*description of bundle* $\mathcal{E}$
1)	*0*	*0*	$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$
2)	*1*	*0*	$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$
3)	*1*	*1*	$\mathcal{E} = T\mathbf{P}^2(-1)$
4)	*2*	*0*	$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$
5)	*2*	*1*	$\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$
6)	*2*	*2*	$0 \to \mathcal{O} \to T\mathbf{P}^2(-1) \oplus \mathcal{O}(1) \to \mathcal{E} \to 0$
7)	*2*	*3*	$0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{E} \to 0$
8)	*2*	*4*	$0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus 3} \to \mathcal{E} \to 0$

Fano Mori Contractions of a 4-fold with a fiber $F = \mathbf{P}^2$.

Take now the *total space of* $\mathcal{E}$ and the map φ associated to the evaluation of sections (or its projective closure $\mathbf{P}(\mathcal{E} \oplus \mathcal{O})$ and the map associated to the tautological bundle).

This map is a F-M contractions which contracts the zero section $F_0 = \mathbf{P}^2$ to a point and the conormal bundle is $\mathcal{E}$.

It is described for each case in the following:

1) $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$, then φ is a scroll, Z is smooth

2) $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$, then φ is a smooth blow-up of a smooth curve, Z is smooth

3) $\mathcal{E} = T\mathbf{P}^2(-1)$, then φ is a generalized scroll, i.e. a one dimensional scroll with a two dimensional fiber, Z is smooth

4) $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$, then φ is the blow-up of a smooth curve which is $= SingZ$

5) $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$, then φ is the small contraction, Z is singular and the flip exists.

6) $0 \to \mathcal{O} \to T\mathbf{P}^2(-1) \oplus \mathcal{O}(1) \to \mathcal{E} \to 0$, then φ is the blow-up of a smooth surf. passing from a quadric sing. of Z

7) $0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{E} \to 0$, then φ is the blow-up of a surface with a singular point of the type of a cone over a twisted cubic in a smooth Z

8) $0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus 3} \to \mathcal{E} \to 0$, then φ is a conic bundle with a two dimensional fiber, Z is singular.

F- M Contractions of a 4-fold with a fiber a two dim. quadric.

Let F be a quadric, i.e. $F = \mathbf{F}_0$ or $\mathbf{S}_2$ or $\mathbf{P}^2 \cup \mathbf{P}^2$; let $\mathcal{O}(1)$ be the line bundle on F which is the restriction of $\mathcal{O}(1)$ from $\mathbf{Q}^4$.

By the above results $\mathcal{E} = N^*_{F/X}$ is spanned by global sections and, by adjunction formula, $0 \leq c_1(\mathcal{E}) \leq 1$.

Proposition. *Let $\mathcal{E}$ be a rank-2 vector bundle over a 2-dim. quadric V. Suppose that $\mathcal{E}$ is spanned by global sections and that $c_1(\mathcal{E}) = \mathcal{O}(1)$. Then, up to an automorphism of the quadric, we have that E is either*

(1) $\mathcal{O} \oplus \mathcal{O}(1)$, *or*

(2) a pull-back of $T\mathbf{P}^2(-1)$ *via a double covering* $V \to \mathbf{P}^2$ *or*

(3) $\mathcal{S}(1)$, *where* $\mathcal{S}$ *is the Spinor bundle.*

Let $\mathbf{Q}_4 \simeq Gr(1,3)$ be the smooth 4-dimensional quadric, identified with the Grassmaniann of lines in $\mathbf{P}^3$ and let $\mathcal{S}$ be the universal bundle which we call also *the spinor bundle* on $\mathbf{Q}_4$.

Let V be a (codimension 1 or 2) linear section $i : V \hookrightarrow \mathbf{Q}_4$. Call, by abuse of notation, $\mathcal{S} = i^*(\mathcal{S})$ *the spinor bundle.*

When $dim V = 2$ we have the following description:

-if $V \simeq \mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, $\mathcal{S} = \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)$;

-if $f : \mathbf{F}_2 \to \mathbf{S}_2 = V$, $0 \to \mathcal{O}(-f) \to f^*\mathcal{S} \to \mathcal{O}(f - C_0) \to 0$;

-if $V \simeq \mathbf{P}^2 \cup \mathbf{P}^2 = P_1 \cup P_2$, $\mathcal{S}_{|P_1} = T\mathbf{P}^2(-2)$ and $\mathcal{S}_{|P_2} = \mathcal{O} \oplus \mathcal{O}(-1)$.

Complete Intersections in Projective Bundles

Consider a point $v \in \mathbf{P}^4$ and the following incidence diagram

$M = \{(x, \mathbf{P}^2) \in \mathbf{P}^4 \times \mathbf{Q}_4 : x, v \in \mathbf{P}^2 \subset \mathbf{P}^4\} \quad \longrightarrow \mathbf{P}^4$

$\downarrow$

$\mathbf{Q}_4 = \{\mathbf{P}^2 : v \in \mathbf{P}^2 \subset \mathbf{P}^4\}.$

- $p : M := \mathbf{P}(\mathcal{O} \oplus \mathcal{S}(1)) \to \mathbf{Q}^4$ and $\varphi' : M \longrightarrow \mathbf{P}^4$ is the map associated to the linear system $|\xi|$, the relative hyperplane bundle;

- φ' contracts a section $\mathbf{Q}_0 = \mathbf{Q}_4$ to the point $v \in \mathbf{P}^4$; the normal bundle of $\mathbf{Q}_0$ in M is equal to $\mathcal{S}(1)$.

Take X to be a complete intersection of two divisors in the linear system

$$\mathcal{L} := \varphi'^*(\mathcal{O}_{\mathbf{Q}_4}(1)) + p^*(\mathcal{O}_{\mathbf{Q}_4}(1)).$$

Let $\varphi : X \to \mathbf{P}^4$ be the restriction of φ'.

$F := X \cap Q_0$ is a linear section of $\mathbf{Q}_4 \simeq \mathbf{Q}_0$ and it is a two dimensional fiber of φ; it can be either $\mathbf{F}_0$ or $\mathbf{S}^2$ or $\mathbf{P}^2 \cup \mathbf{P}^2$

Proposition. *Each of these cases can be realized so that the complete intersection X is smooth.*

In coordinates.

$\{(z_0, z_1, z_2, z_3)\} = \mathbf{C}^4 \subset \mathbf{P}^4$
$\{[t_{01}, t_{02}, t_{03}, t_{12}, t_{13}, t_{23}]\} = \mathbf{P}^5$ (Plücker coordinates)

In these coordinates $M \cap \mathbf{C}^4 \times \mathbf{P}^5$ is written as follows

$$\begin{gathered}
g_1 = t_{01}z_2 - t_{02}z_1 + t_{12}z_0 = 0 \\
g_2 = -t_{01}z_3 + t_{03}z_1 - t_{13}z_0 = 0 \\
g_3 = t_{02}z_3 - t_{03}z_2 + t_{23}z_0 = 0 \\
g_4 = -t_{12}z_3 + t_{13}z_2 - t_{23}z_1 = 0 \\
q = t_{01}t_{23} - t_{02}t_{13} + t_{12}t_{03} = 0.
\end{gathered}$$

Two divisors in the linear system $\mathcal{L}$ are given by the equations (this is a family depending from 3 parameters (p, r, s)).

$$\begin{aligned}
f_1 = {} & t_{01} - pt_{23} & + & \quad z_1t_{12} + z_2t_{23} + rz_3t_{13} \\
f_2 = {} & t_{02} - st_{13} & + & \quad z_0t_{03} + z_3t_{13}
\end{aligned}$$

$X = V(f_i, g_i, q)$ is smooth by the Jacobian criterion.

On $\mathbf{Q}_0$ these equation cut out a net of quadrics:

–$ps \neq 0$ this is a smooth quadric.

–for $ps = 0$ this is a quadric cone, while for

–for $p = s = 0$ the intersection is a reducible quadric, The degeneracy locus of the two sections in $\varphi'_*\mathcal{L}$ will give the variety to blow-up in $\mathbf{C}^4 \subset \mathbf{P}^4$ to get the contraction φ. Using general formulas, one can get that it is a surface singular in v of degree 9, and that the ideal I_S is defined by six quartics,..

In coordinates one can just look at the previous equations in the variables $t_{i,j}$ with coefficients in the variable z_i

The matrix of coefficients of this system is

$$M := \begin{pmatrix} z_2 & -z_1 & 0 & z_0 & 0 & 0 \\ -z_3 & 0 & z_1 & 0 & -z_0 & 0 \\ 0 & z_3 & -z_2 & 0 & 0 & z_0 \\ 0 & 0 & 0 & -z_3 & z_2 & -z_1 \\ 1 & 0 & 0 & z_1 & r \cdot z_3 & z_2 - p \\ 0 & 1 & z_0 & 0 & z_3 - s & 0 \end{pmatrix}$$

I_S is the ideal of the minors 5×5 of the matrix M.

Using *CoCoa* we found the following generators of the ideals:

$$\begin{aligned}
h_1 &:= z_0 z_3^3 r + z_0 z_1 z_2 z_3 + z_2 z_3^2 r + z_1 z_2^2 - z_0^2 z_3 - z_1 z_3^2 + z_1 z_3 s - z_0 z_2 \\
h_2 &:= -z_0 z_1^2 z_3 + z_0 z_2 z_3^2 - z_0 z_3^2 p - z_1^2 z_2 + z_2^2 z_3 - z_2 z_3 p + z_0 z_1 \\
h_3 &:= -z_0 z_1 z_3^2 r - z_0 z_2^2 z_3 - z_1 z_2 z_3 r + z_0 z_2 z_3 p + z_0^2 z_1 - z_2^3 + z_1^2 z_3 - \\
&\quad z_1^2 s + z_2^2 p \\
h_4 &:= z_1^2 z_3^2 - z_2 z_3^3 - z_0 z_3^2 r - z_1^2 z_3 s + z_2 z_3^2 s + z_3^3 p - z_3^2 sp - z_0 z_1 z_2 + z_0^2 \\
h_5 &:= -z_0^2 z_3^2 r - z_0^2 z_1 z_2 z - z_1^2 z_2 z_3 + z_2^2 z_3^2 + z_1^2 z_2 s - z_2^2 z_3 s - z_2 z_3^2 p + \\
&\quad z_2 z_3 sp + z_0^3 + z_0 z_1 z_3 - z_0 z_1 s \\
h_6 &:= z_0^2 z_1^2 + z_1^3 z_3 - z_0^2 z_2 z_3 - z_1 z_2 z_3^2 - z_0 z_1 z_3 r - z_1^3 s + z_1 z_2 z_3 s + \\
&\quad z_0^2 z_3 p + z_1 z_3^2 p - z_1 z_3 sp - z_0 z_2^2 + z_0 z_2 p
\end{aligned}$$

Problem: When is the formal or the analytic neighbourhood of F in X uniquely determined by $N^*_{F/X}$?

Proposition. (Grauert; Hironaka-Rossi). *If*

$$H^1(F, T_X \otimes S^k(N^*)) = 0 \ \text{ for all } \ k > 0$$

it is uniquely determined.
In particular if F is a smooth and $N^* = N^*_{F/X}$ is nef then if

$$H^1(F, T_F \otimes S^i(N^*) = H^1(F, N \otimes S^{i+1}(N^*)) = 0 \ \textit{ for } i \geq 1$$

the analytic neighbourhood of F in X is uniquely determined.

Corollary. *In the above example if the fiber is $\mathbf{F}_0$ then the analytic neighborhood is uniquely determined.*

Remark . If the fiber is $\mathbf{P}^2 \cup \mathbf{P}^2$ then the analytic neighborhood is not unique (althought the target is smooth).
To see this compute the tangent cone of S in $(z_i = 0)$; for $s = p = 0$ (the special fiber of φ is a reducible quadric) it is generated by the following functions

$$(z_0^2, \ z_0 z_1, \ z_0 z_2, \ -z_1 z_2 z_3 r - z_2^3 + z_1^2 z_3)$$

which defines a scheme structure over a plane rational cubic

$$z_0 = z_1 z_3 (z_1 - r z_2) - z_2^3 = 0$$

with an embedded point at $z_0 = z_1 = z_2 = 0$ which is also the singularity of the cubic. For $r \neq 0$ the singularity is a node while for $r = 0$ it degenerates to a cusp.

How to explain this last fact:

Assume $F = F_1 \cup F_2 = \mathbf{P}^2 \cup \mathbf{P}^2$; $N_{F_2/X}$ and $N_{F_1/X}$ can be recovered from $N_{F_1 \cup F_2/X} \simeq \mathcal{S}$ via a "vector bundle surgery" which gives a unique bundle also if we start from different $N_{F_1/X}$.

The Hilbert scheme of lines in fiber.

The general fiber of $\varphi_{|E}$ is a line (w.t.r. to $-K_X$) with the normal $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)$.

Suppose that F_1 is a component of the fiber F.

A line $l \subset F_1$ *moves out* of F_1 if and only if it is a jumping line of the bundle $N_{F_1/X}$, which means that the restriction of the latter bundle to l is either $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$.

Applications ...

Let $\varphi : X \to Z$ be a proper birational morphism between **smooth** varieties. Put $S := \{z \in Z | dim\varphi^{-1}(z) \geq 1\}$ and $E = \varphi^{-1}(S)$; assume that $(E)_{red}$ is an irreducible divisor.

Proposition. *If $E \to S$ is equidimensional then φ is the smooth blow up; i.e. S is smooth and φ is the blow-up of Z along S.*

Proposition. *If the dimension of $S \leq 2$ (and S is not smooth) then $dimX = 4$ and φ is the blow-up of a non normal surface with isolated singularities. The fibers over the singular points have dimension two and together with their normal bundle they were described above.*

More generally one should prove

Proposition. *If the dimension of the special fiber of $E \to S$ is less or equal to the dimension of the general fiber plus one......*