Recent results on Fano manifolds

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Riemann-Poincaré uniformization theorem

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Let X be a complex projective manifold:

If $dimX := n = 1$ then there exits an hermitiam metric on TX with constant curvature k such that

$k > 0$	$k = 0$	$k < 0$
$X = S^2 = \mathbb{P}^1$	$X = \mathbb{C}/\Gamma$	$X = \Delta/\pi_1(X)$

And in higher dimension ?
How can we generalize the first class?

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- TX is not generically seminegative
  i.e. $\exists \mathcal{E} \hookrightarrow TX$ and $\{C_t\}$ a family of curves

  such that $c_1(\mathcal{E})C_t > 0$ and $\{C_t\}$ covers X
  $\sim \exists f : \mathbb{P}^1 \to X$ and f^*TX is nef.

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Minimal Model Program

$X \sim$ fibrazione in varietá di Fano

I Maestri

Rational curves

On uniruled varieties we have families of rational curves, i.e. irreducible components
$V \subset \mathrm{Ratcurves}^n(X) := \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)/\mathrm{Aut}(\mathbb{P}^1)$
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Deformation theory+Rieman-Roch give a bound to the dimension from below.
It works very well for Fano manifolds:

$$dim V \geq -K_X \cdot C + (n-3),$$

$$dim V_x \geq -K_X \cdot C - 2$$

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Remark. If V is gen unsplit then:

$$dimLocus(V_x) = dimV_x + 1 \geq -K_X{\cdot}C - 1.$$

Mori bend and break

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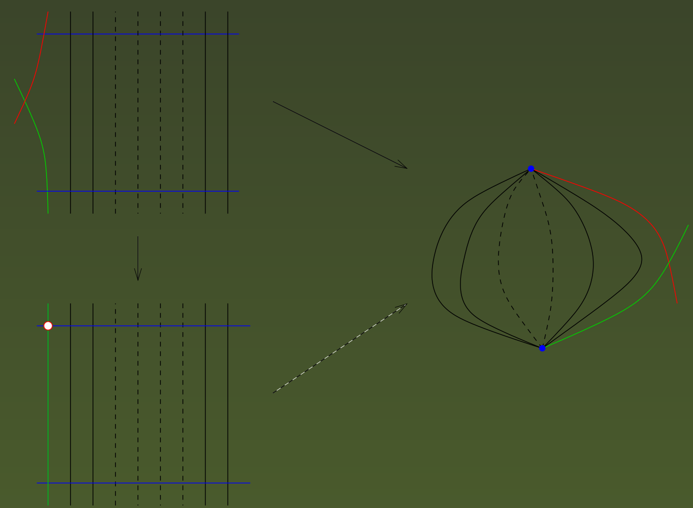

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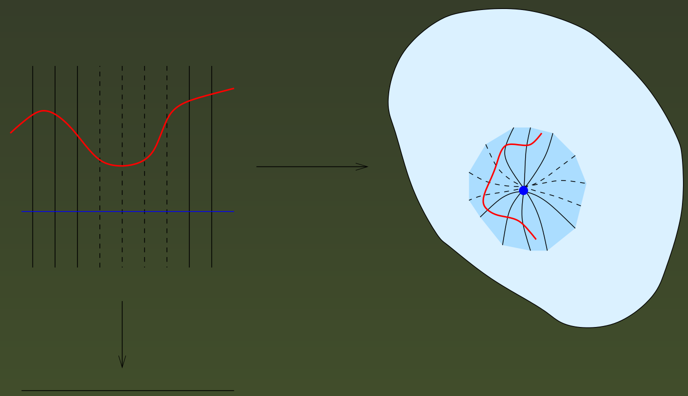

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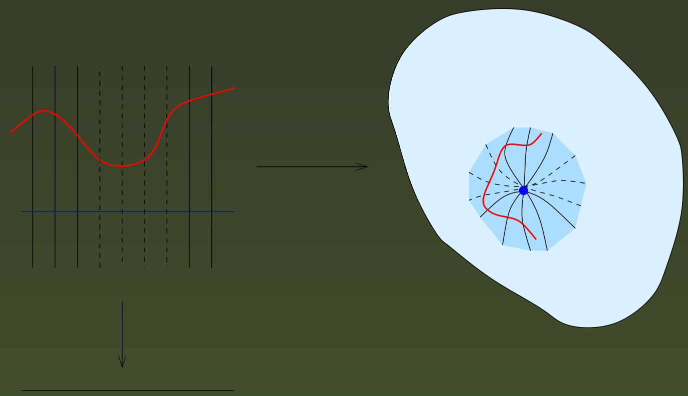

Proposition Let V be an unsplit family and $Y \subset X$ a closed subset such that every curve in Y is independent from curves in V. Then

$$dimLocus(V)_Y \geq dimY + deg_{-K_X}V - 1.$$

Rationally connected fibrations.

Let $x, y \in X$ and define :
$x \sim y$ iff $\exists$ a chain of rational curves through x and y.
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Theorem- Campana and Kollár-Miyaoka-Mori (1992)
The exists an open set $X^0 \subset X$ and a map $\varphi^0 : X^0 \to Z^0$ which is proper, with connected fiber and whose fibers are equivalence classes for the equivalence relation $\sim$ (therefore they are rationally connected).
If V is unsplit the same is true for $\sim_{rcV}$ (and the fibers are rationally connected with respect to V).

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Conjecture of Mukai (1988):

$$\rho_X(r_X - 1) \leq n.$$

later **generalized**

$$\rho_X(i_X - 1) \leq n \text{ with} = \text{iff} \quad X \simeq (\mathbb{P}^{i_X - 1})^{\rho_X}.$$

Steps toward the conjecture

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-(2003) Andreatta,Chierici,Occhetta:
G.C. holds if (a) $n = 5$,
(b) if $i_X \geq \frac{n+3}{3}$ and there exists a family of rational curves V which is unsplit and covers X.
This is the case if X has a fiber type contraction or it does not have small contractions.

A naive proof of the conjecture

Proposition [Wi] Let V be an unsplit family and $Y \subset X$ a closed subset such that every curve in Y is independent from curves in V. Then

$$dim Locus(V)_Y \geq dim Y + deg_{-K_X} V - 1.$$

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Claim. If there exist $V_1, ..., V_\rho$ unsplit families of r.c. whose classes are lineraly independent in $N_1(X)$ and such that $Locus(V_1, ..., V_\rho)_x \neq \emptyset$ then the conjecture holds.

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It is enough to apply inductively the proposition:

$$n \geq dimLocus(V_1, ..., V_\rho)_x \geq \Sigma_j(degV_j - 1) \geq \rho(i_X - 1).$$

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For other steps one has to put three or more families ”in row”...

Classification (with high r_X or i_X).

X Fano manifold

X singular: M. Reid and his school.

$r_X \geq n+1$	$\mathbb{P}^n$	Kobayashi-Ochiai (70)
$i_X \geq n+1$	$\mathbb{P}^n$	Cho-Miyaoka-Sh.Barr.(01)

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$r_X = n-1$	del- Pezzo	Fujita (90)
$r_X = n-2$	Mukai	Fano, Isk., Mori, Mukai, ...

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If $r_X = (n-1)$ then an elephant exists (Mella 99).

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The last discovered Fano 3-fold (Mori-Mukai 2003):
$Bl_C(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1), C \text{ curve of three} - \text{degree } (1,1,3)$

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Time to ”ascend”. The adjunction theory studies, in more generality, the following problem:

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Remark This is not a problem if we start by X.

Tell me which cone you have and I'll tell who you are

Theorem Andreatta-Novelli-Occhetta (2003).
If $r_Z \geq dim Z/2 \geq 2$ then
X is Fano and $\overline{NE(Z)} = \overline{NE(X)}$,
except if $Z = \mathbb{P}^1 \times V$, with $V = \mathbb{P}^3$ a del Pezzo 3-fold
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Remark:

1) if L is very ample the theorem is by Beltrametti-Fania-Sommese

2) The last discvered Fano 3-fold does not ascend

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Theorem Chierici-Occhetta (2004).
Let X be a Fano manifold with $i_X = dimX - 3$, $dimX \geq 4$ (and $\rho_X \geq 2$). The cones $\overline{NE(X)}$ are listed for all possible X.

In particular $\overline{NE(X)}$ is generated by ρ_X rays.

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In particular $\overline{NE(X)}$ is generated by ρ_X rays.

Moreover X has always an elementary fiber type contraction except when:
X is the blow up of $\mathbb{P}^5$ along one of the following surfaces: a smooth quadric, a cubic scroll in $\mathbb{P}^4$, a Veronese surface.