Recent results on Fano manifolds

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Positivity of the tangent bundle

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- $\mathcal{E} \subset TX$ ample subbundle iff $X = \mathbb{P}^n$, Andreatta-Wisniewski '01.

- TX is not generically seminegative
i.e. $\exists \mathcal{E} \hookrightarrow TX$ and $\{C_t\}$ a family of curves such that $c_1(\mathcal{E})C_t > 0$ and $\{C_t\}$ covers X
iff X is uniruled Miyaoka '87.

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The if part was proved by Kollár-Miyaoka-Mori,
the only if is in the Minimal model conjecture.

Rational curves

On uniruled varieties we have families of rational curves, i.e. irreducible components

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Deformation theory+Rieman-Roch give a bound to the dimension from below: let C be a curve in V

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Note that it works well for Fano manifolds

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Remark. If V is gen unsplit then:

$dim Locus(V_x) = dim V_x + 1 \geq -K_X \cdot C - 1.$

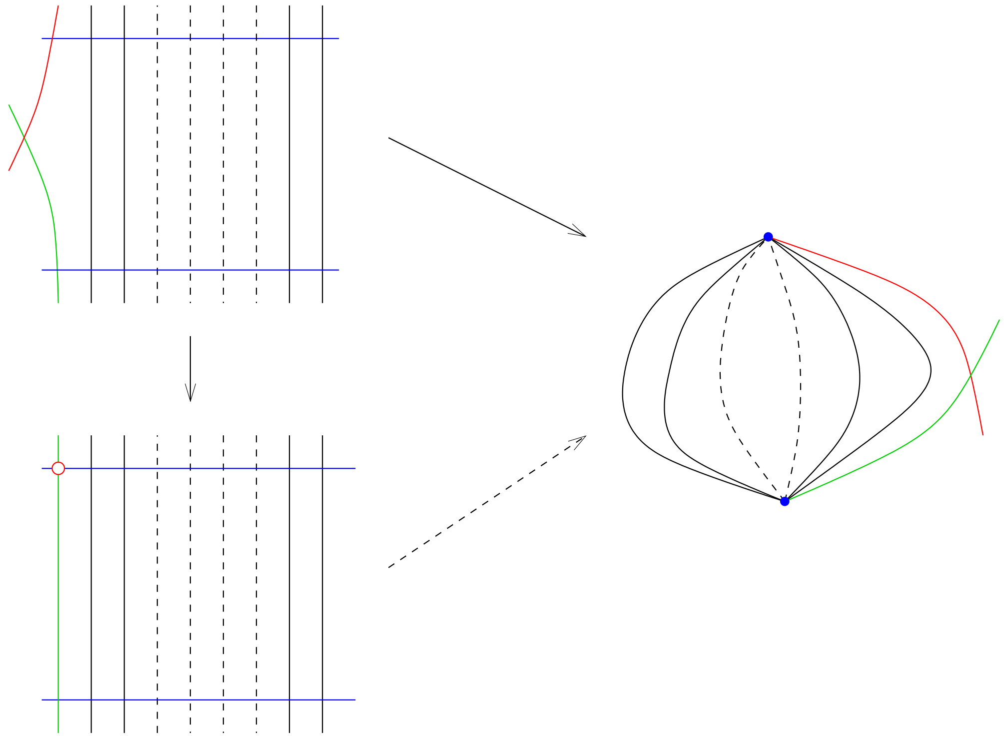
Uniruledness of Fano manifolds

Theorem [Mori] and [Kollár-Mori-Miyaoka]
Let X be a Fano manifold (and let $\pi : X^0 \to Z^0$ be a proper surjective morphism).
There exists a family V of rational curves such that

- $-K_X \cdot V \leq \dim X + 1$
- V is generically unsplit
- for $z \in Z^0$ generic there exists a C in V such that $C \cap \pi^{-1}(z) \neq \emptyset$ and C is not contained in $\pi^{-1}(z)$ (covering family).

Mori - bend and break

Proof of generically unsplittedness :



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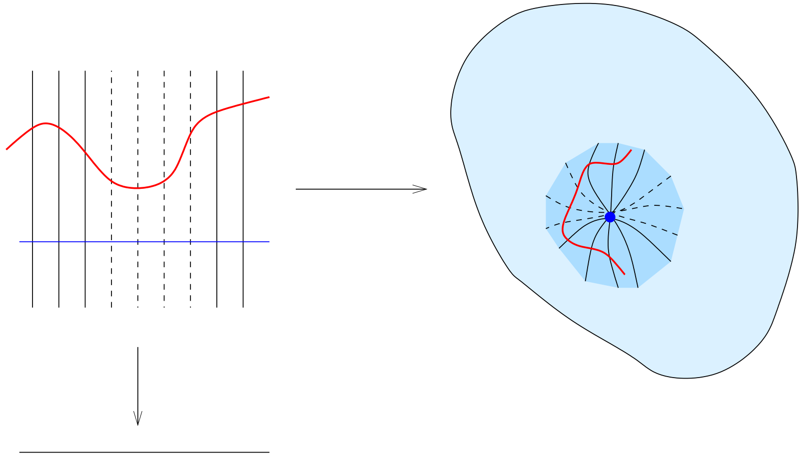

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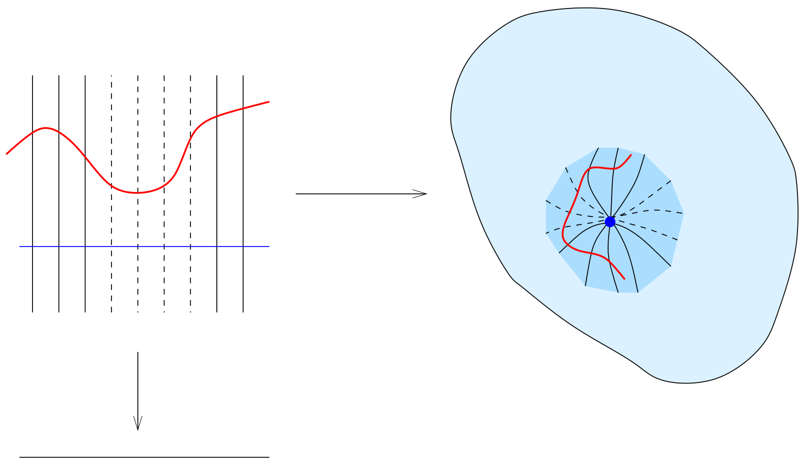

Proposition Let V be an unsplit family and $Y \subset X$ a closed subset such that every curve in Y is independent from curves in V. Then

$$dimLocus(V)_Y \geq dimY + deg_{-K_X}V - 1.$$

Rationally connected fibrations.

Let$X$ be uniruled, $x, y \in X$ and define :
$x \sim y$ iff $\exists$ a chain of rational curves through x and y.
$x \sim_{rcV} y$ iff $\exists$ a chain of rational curves $\in V$ through x and y

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Theorem [Campana] and [Kollár-Miyaoka-Mori] (1992)
The exists an open set $X^0 \subset X$ and a map $\varphi^0 : X^0 \to Z^0$ which is proper, with connected fiber and whose fibers are equivalence classes for the equivalence relation $\sim$ (therefore they are rationally connected).
If V is unsplit the same is true for $\sim_{rcV}$ (and the fibers are rationally connected with respect to V).

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Conjecture of Mukai (1988):

$$\rho_X(r_X - 1) \leq n.$$

later **generalized**

$$\rho_X(i_X - 1) \leq n \ \text{ with} = \text{iff} \quad X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}.$$

Steps toward the conjecture

-(1990) Wiśniewski:
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-(2002) Bonavero, Casagrande, Debarre e Druel:
G.C. holds if (a) $n = 4$,
(b) X is toric and $i_X \geq \frac{n+3}{3}$ or $n \leq 7$.

Steps toward the conjecture

-(2003) Andreatta,Chierici,Occhetta:
G.C. holds if (a) $n = 5$,
(b) if $i_X \geq \frac{n+3}{3}$ and there exists a family of rational curves V which is unsplit and covers X.
This is the case if X has a fiber type contraction or it does not have small contractions.

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-(2004) Casagrande:
G.C. holds for toric varieties

A naive proof of the first part

Proposition [Wi] Let V be an unsplit family and $Y \subset X$ a closed subset such that every curve in Y is independent from curves in V. Then

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Claim. If there exist $V_1, ..., V_\rho$ unsplit families of r.c. whose classes are lineraly independent in $N_1(X)$ and such that $Locus(V_1, ..., V_\rho)_x \neq \emptyset$ then the conjecture holds.

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It is enough to apply inductively the proposition:

$$n \geq dimLocus(V_1, ..., V_\rho)_x \geq \Sigma_j(degV_j - 1) \geq \rho(i_X - 1).$$

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Theorem (Wiśniewski 90)
If $i_X > \frac{n+2}{2}$ and $\rho_X > 1$ then there exist two families V_1, V_2 as in the claim. In fact:

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Since $\rho_X > 1$ there must be another family whose curves are independent (cone theorem).

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In general one try to start with an unsplit dominant family V and construct the rcV-fibration.
If the dimension of the target is zero (i.e. X is rationally V-connected) then $\rho = 1$.

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If not there exists a locally unsplit family V' which is transverse and dominant with respect to the rcV-fibration.
By our assumption $i_X > \frac{n+3}{3}$, also this family is unsplit.

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Construct the $rc(V, V')$-fibration. If the dimension of the target is zero then $\rho = 2$.

....

The second part of the conjecture

Theorem [Cho-Miyaoka-Sh.Barron] - [Kebekus]
If there exists a family of rational curves V of degree $dimX + 1$, unsplit (and dominant, i.e. $Locus(X) = X$) then X is the projective space.

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Theorem [G. Occhetta]
If there exist k families of rational curves which are unsplit, dominant, independent in $N_1(X)$ and whose sum of degree minus k is equal to $dimX$ then X is the product of k projective spaces.

The second part, proof

The key point is to prove that, for x generic, the natural derivative map (Mori),

$$\Phi_x : V_x^n \to P(T_x X) = P((f^* TX)_0)$$

$$\Phi_x([f]) = [(df)_0(\partial/\partial t)]$$

is birational.

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- Φ_x is a regular map and it is finite (Kebekus).
- Φ_x is surjective (this is clear if $-K_X \cdot C \geq (n+1)$)
- if Φ_x is surjective then it is birational and thus a biholomorphism (a lemma of Miyaoka says that if Φ_x is not generically injective then there exists a curve in V singular at x).

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2	del Pezzo
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$Bl_C(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1), C$ curve of $3 -$ degree $(1, 1, 3)$

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By induction, or adjunction, restricting to an elephant = element of the anticanonical system i.e. a smooth section $D \in L$ where $-K_X = r_X L$, whose existence is by Fujita-90 ($r_X \geq (n-1)$) and Mella-99 ($r_X = (n-2)$).

$\rho_X \geq 2$, *the two rays game*

If $\rho_X \geq 2$ the right invariant is the pseudondex i_X.
For instance $X = \mathbb{P}^n \times \mathbb{P}^{n+1}$ has $r_X = 1$ and $i_X = n + 1$.

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Theorem [Andreatta-Occhetta (2004)]. If $\rho_X \geq 2$
$(2i_X \leq)\ \ l(R) + i_X \leq \dim Exc(R) + 2$.

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$(2i_X \leq) \;\; l(R) + i_X \leq \dim Exc(R) + 2$.

If equality holds and R is not small then
$X \simeq \mathbb{P}^k \times \mathbb{P}^{n-k}$ or $X \simeq Bl_{\mathbb{P}^k}(\mathbb{P}^n)$ with $k \leq \frac{n-3}{2}$

$\rho_X \geq 2$, *the blow-ups*

Let X be the the blow up of a manifold Y along $T \subset Y$, and let $i_X \geq \dim T + 1$ (i.e. $l(R) + i_X = \dim Exc(R) + 1$).
Then X is one of the following

1. $Bl_p(\mathbb{P}^n)$.
2. $Bl_p(\mathbb{Q}^n)$.
3. $Bl_p(V_d)$ where V_d is $Bl_Y(\mathbb{P}^n)$ and Y is a submanifold of dimension $n-2$ and degree $\leq n$ contained in an hyperplane.
4. The blow up of $\mathbb{P}^n$ along a $\mathbb{P}^k$ with $k \leq \frac{n}{2} - 1$.
5. $\mathbb{P}^1 \times Bl_p(\mathbb{P}^{n-1})$.
6. The blow up of $\mathbb{Q}^n$ along a $\mathbb{P}^k$ with $k \leq \frac{n}{2} - 1$.
7. The blow up of $\mathbb{Q}^n$ along a $\mathbb{Q}^k$ with $k \leq \frac{n}{2} - 1$.

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Theorem [Chierici-Occhetta (2004)].
Let X be a Fano manifold with $i_X = dimX - 3$, $dimX \geq 5$ and $\rho_X \geq 2$.
The cones $\overline{NE(X)}$ are listed for all possible X. In particular it is generated by ρ_X rays.

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Moreover X has always an elementary fiber type contraction except when:
X is the blow up of $\mathbb{P}^5$ along one of the following surfaces: a smooth quadric, a cubic scroll in $\mathbb{P}^4$, a Veronese surface.