Note for a conference

Actions of Linear Algebraic Groups on Projective manifolds

Marco Andreatta

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Set up. In this talk X will be a smooth projective variety of dimension n on which a linear connected algebraic group G acts algebraically and non trivially.
We will actually assume that G is semisimple and thus we may also assume that G is simply connected (via the universal cover) (note that once we assume the existence of a closed orbit of dimension > 0 then G is semisimple).

General Problem Fixed G, classify such X.

Special case: **Homogeneous manifolds.**
These are completely classified and they correspond to the parabolic subgroups which in turn correspond to a choice of a subset of nodes of the Dynkin diagram.
Note that an homogeneous manifold is a rational Fano manifold ($-K_X$ is ample and T_X is generated by g.s.) and that they are fundamental examples in various aspects of Alg. Geom. (for instance Severi varieties, i.e. variety in $\mathbf{P}^m$ of dimension $\geq 2/3(m-2)$) which are embedded into $\mathbf{P}^{m-1}$ by the projection of a general point).

The approach at the general problem we choose is to start with projective varieties of small dimension with respect to G, i.e. with too many symmetries; to state this properly we introduce a definition.
Definition. Let G be a simple (or even semisimple), simply connected and connected Lie group G. We define $r = r_G$ to be the minimal codimension of parabolic subgroups of G. We will call the homogeneous variety associated to this parabolic subgroup a minimal homogeneous variety for the action of G.
Note that $r_{SL(m)} = r_{Sp(m)} = m-1$ with $G/P = \mathbf{P}^{(n-1)}$, $r_{Spin(m)} = m-2$ with $G/P \simeq Q^{(m-2)} \subset \mathbf{P}^{(m-1)}$ and $r_{G_2} = 5, r_{F_4} = 15, r_{E_6} = 16, r_{E_7} = 27, r_{E_8} = 57$ (if $G = E_6$ the minimal h.v. is the fourth Severi variety in the Zak's theorem).

Proposition. *Let X and G be as in the Set-up (more generally don't even assume that X is smooth). Then $n \geq r_G$ and if $n = r_G$ then X is homogeneous (the minimal h. v.). For instance if $G = SL(m)$ this says that if G acts non trivially on a projective variety of dimension $\leq (m-1)$ then $X = \mathbf{P}^{m-1} = SL(m)/P$.*

Remark . If $n = 3$ this was the main results in Nakano, NagoyaJ. '89.

Proof. The proof follows immediately from the following two facts.
a) Closed orbits exist (Borel).
b) Lemma. If $n \leq r_G$ there are no fixed points.

Proof of the lemma a). The assumption implies that there are no irreducible representation of dimension $> n$. Thus if x is a fixed (smooth) point G acts as the identity on T_xX and this gives the lemma, for instance via the following Luna's slice theorem.

Local Linearization Theorem [Lu]. *Suppose that a reductive group G acts on a smooth quasiprojective variety X with a fixed point $z \in X$. Then there exists an open affine G-invariant neighborhood $z \in S \subset X$ and an étale G-equivariant map $S \to T_zX$.*

Any stronger result on fixed points will be very useful and we will use the following:

Proposition of the cone. *Suppose that a connected reductive linear algebraic group G acts effectively on a complete normal variety Z. Then the followings are equivalent:*
a) There exists a fixed point z such that $P_z = Proj[gr(\mathcal{O}_{Z,z})]$ is a G-homogeneous variety.
b) Z has an open orbit Ω and $A := Z \setminus \Omega$ contains an isolated point z.
c) Z is a projective quasi homogeneous cone over a homogeneous variety P with the respect to G.

Proof. a) implies b): z is contained in the closure of a non-trivial G-orbit (not obvious!). The tangent cone of this orbit and of its closure is G invariant. Since by assumption P_z is G-homogeneous, this implies that the orbit has dimension equal to the dimension of Z.
b) implies c): this is a theorem of Ahiezer.
c) implies a): this is obvious.

Remark . We can prove that a) implies c) directly without using Ahiezer result but making use of the Borel-Weil-Bott theory.

Corollary. *If $n = r_G + 1$ then X has no smooth fixed points unless $G = SL(n)$ or $Sp(n = 2s)$ and $X = \mathbf{P}^n$ and the action is induced from the homomorphism $g \to (1, 0; 0, g)$ from $SL(n)$ or $Sp(n)$ to $PGL(n + 1)$).*

Question. Let Z be a normal variety on which a reductive linear algebraic group operates with an open orbit. If z is a fixed point then does Z contain a Zariski open subset G-equivariant to the cone over z?

A possible approach *is the following: given a group G find an inductive procedure to classify the projective varieties X as in the set up starting with the case $n = r_G$ and increasing n; the inductive step will be determined via a MMP construction.*

Some definitions and fundamental facts from the Minimal Model Program (MMP).

Definition. A projective variety X with good singularities, in particular smooth, will be said minimal (in the sense of MMP) if K_X is nef.

Definition-Theorem (Mori-Kawamata-Shokurov-...). Let X be a smooth variety which is not minimal. Then there exists a map $\varphi : X \to Z$ into a normal projective variety Z with connected fibers such that $-K_X$ is φ-ample and φ contracts the set of curves numerically equivalent to a (rational) curve in a non trivial fiber. Such a map is called an extremal contraction or a Fano-Mori contraction.

... and how they fit in our Set-up.

Lemma 1. *Let X and G be as in the Set-up. Then X is uniruled and therefore it is not minimal.*

Proof. On the generic point the action is not trivial, hence it is contained in an orbit which is unirational since G is rational. Thus the generic point is contained in a rational curve $\subset X$.

Lemma 2. *There exists a Fano-Mori contraction on X, $\varphi : X \to Z$, which is G-equivariant and G operates on Z.*

Proof. Since X is not minimal it has a Fano-Mori contraction φ and since two curves G equivariant are numerically equivalent φ is equivariant.

Running the MMP in the case $n = r_G + 1$ is successufully

By the above Lemmas there exists a G-equivariant extremal contraction $\varphi : X \to Z$. We consider the special case $G = SL(n)$ and we have

a) φ is birational, then it contracts a divisor to a point z and the divisor has to be $\mathbf{P}^{(n-1)}$. Therefore Z is G-isomorphic to the cone over z and thus X is $X_m = \mathbf{P}(\mathcal{O}_{\mathbf{P}^{(n-1)}}(m) \oplus \mathcal{O}_{\mathbf{P}^{(n-1)}})$ or, if $m = 1$, we have also $\mathbf{P}^n$ (which has however $\rho = 1$).

b) φ is of fiber type, then

i) either $Z = \mathbf{P}^{(n-1)}$ and φ is an equivariant $\mathbf{P}^1$-bundle, (the action on Z is homogeneous and thus the fibers are all equidimensional and there are no reducible or double fibers). Thus $X = \mathbf{P}(E)$ with E an homogeneous rank 2 vector bundle on Z (since the action is φ equivariant). That is, after normalizing if necessary, $E = \mathcal{O}(s) \oplus \mathcal{O}$ with $s \geq 0$ or $n = 3$, $G = SL(3)$ and $E = TZ$.
If $E = \mathcal{O}(s) \oplus \mathcal{O}$ we have a decomposition of X into three orbits. Two isomorphic to $\mathbf{P}^{(n-1)}$ (the sections at infinity and the zero section) and a open dense orbit isomorphic to G/S where S is the kernel of the character map $\rho : H \to \mathbf{C}^*$ associated to the homogeneous line bundle $\mathcal{O}(s)$. Therefore $S =$......
If $n = 2$, $s = 0$ and $G = SL(2)$ then we have an other case which comes from the diagonal action of $SL(2)$ on $\mathbf{P}^1 \times \mathbf{P}^1$.
If $n = 3$ and $E = T\mathbf{Z}$ it is well known that $X = \mathbf{P}(T\mathbf{P}^2)$ is the homogeneous variety G/B where B is the Borel subgroup of $SL(3)$.

ii) Z is a curve with a trivial action; then $X = \mathbf{P}^{(n-1)} \times Z$

ii) Z is a point, i.e. X is Fano with $\rho(X) = 1$.
If X is homogeneous we have only one possibility for $n = 4$, namely $X = SL(4)/Q$ where Q is the parabolic subgroup corresponding to the second node of the Dynkin diagram A_4, i.e.the Grassmanian of planes in $\mathbf{C}^4$, i.e. the smooth 4 dimensional quadric.
If X is not homogeneous and has no fixed point then it must have a closed orbit $H \simeq \mathbf{P}^{(n-1)}$ which is an ample divisor, thus X is isomorphic to $\mathbf{P}^n$ (if $n = 2$ we can have also $\mathbf{P}^1 \times \mathbf{P}^1$, this has however $\rho(X) = 2$ and thus it was considered above).

This and similar arguments for the other classical group (or in general for simple Lie groups) prove the following theorem:

Theorem *Let X be a smooth projective manifold of dimension n and G a Lie group acting non trivially on X.*

If $G = SL(n)$ then X is isomorphic to one of the following varieties; the action of G is unique for each case and it is described in the course of the proof (see also [Ma3]):

1) *the complex projective space $\mathbf{P}^n$,*

2) *$\mathbf{P}^{(n-1)} \times R$, where R is a smooth projective curve,*

3) *The projective bundles $\mathbf{P}(\mathcal{O}_{\mathbf{P}^{(n-1)}}(m) \oplus \mathcal{O}_{\mathbf{P}^{(n-1)}})$ with $m > 0$,*

4) *if $n = 2$ we have an extra action on $\mathbf{P}^1 \times \mathbf{P}^1$ and on $\mathbf{P}^2$,*

5) *if $n = 3$ we have moreover the projective bundle $\mathbf{P}(T(\mathbf{P}^2))$,*

6) *if $n = 4$ we have moreover the smooth 4-dimensional quadric which is isomorphic to $Gr(2, 4)$, the Grassmannian of 2-planes in $\mathbf{C}^4$.*

If $G = Sp(n)$ then X is isomorphic to one of the following varieties and the action of G is unique for each case.

1) *the complex projective space $\mathbf{P}^n$,*

2) *$\mathbf{P}^{(n-1)} \times R$, where R is a smooth projective curve,*

3) *The projective bundle $\mathbf{P}(\mathcal{O}_{\mathbf{P}^{(n-1)}}(m) \oplus \mathcal{O}_{\mathbf{P}^{(n-1)}})$ with $m > 0$,*

4) *if $n = 4$ we have moreover $\mathbf{Q}^4$, the homogeneous variety which is the quotient of $Sp(4)$ by the Borel subgroup (which has two structure of a $\mathbf{P}^1$-bundle over $\mathbf{P}^3$ and over $\mathbf{Q}^3$), $\mathbf{Q}^3 \times R$, where R is a smooth projective curve and the projective bundles $\mathbf{P}(\mathcal{O}_{\mathbf{Q}^3}(m) \oplus \mathcal{O}_{\mathbf{Q}^3})$ with $m > 0$.*

If $G = Spin(n + 1)$ with $n \geq 6$ then X is isomorphic to one of the following varieties and the action of G is unique for each case.

1) *the complex projective space $\mathbf{P}^n$,*

2) *the complex projective quadric $\mathbf{Q}^n \subset \mathbf{P}^{(n+1)}$,*

3) *$\mathbf{Q}^{(n-1)} \times R$, where R is a smooth projective curve,*

4) *The projective bundle $\mathbf{P}(\mathcal{O}_{\mathbf{Q}^{(n-1)}}(m) \oplus \mathcal{O}_{\mathbf{Q}^{(n-1)}})$ with $m > 0$.*

We thus pass to the next cases, namely $n \geq r_G + 2$; but the problem is then too general and one has to assume that X is **quasihomogeneous** ,q.h., i.e. it has an open orbit.

What is known:

- classification of q.h. 3-folds for a $SL(2)$-action; by Mukai-Umemura [M-U] and Nakano [Na1] via Mori's results on 3-folds. In particular Mu-Um found two beautiful examples of Fano 3-folds which are q.h. (forgotten by Fano and Iskovski). For instance, let R_{12} be the vector space of form of degree 12 in two variables with the standard $SL(2)$ action and $f = xy(x^{10} + 11x^5y^5 + y^{10})$; then $X = \overline{SL(2) \circ f}$ is a smooth q.h. 3-fold with Picard number one which is the completion of the open orbit $SL(2)/I$, I being the icosahedral subgroup.

- classification of q. h. 4-folds for an $SL(3)$ action; by Nakano [Na2] with a direct computation of codimension 4 subgroups of $SL(3)$; by myself [An] applying the results of Wisniewski and myself on MMP for 4-folds [A-W].
(let us recall also the results for $\mathbf{C}^*$ actions by M. Reid (i.e. MMP for toric varieties), and more recent one by Brion and by Kebekus...).

- Very beautiful general results for q.h. manifolds are the one of Ahiezer (see [Ah]) (also by M. Brion in the framework of Spherical varieties of rank 1, generalized to the complex analytic case by Huckleberry and Oeljeklaus)
He studied the cases in which the complement of the open orbit is
1) disconnected (two ends)
2) contains an isolated fixed point (we have already quoted this result above)
3) is a homogeneous divisor (this can be handled with the MMP theory).

What is new: in a recent paper with J. Wisniewski we adopt a different approach, based on a theorem of Brion-Luna-Vust, and we proved the following.

Theorem. *Let X be a $SL(n)$ quasihomogeneous manifold and assume $n+1 \leq dimX \leq 2n-2$. Then $dimX \geq 2n-4$ and one of the following occurs:*

(a) $dimX = 2n-4$ and X is homogeneous, $X \simeq Gr(2,n)$;

(b) $dimX = 2n-3$ and either $X = F(1,2,n)$ or X contains a codimension 1 orbit isomorphic to $Gr(2,n)$;

(c) $dimX = 2n-2$ and either $n = 4$ and $X \simeq F(1,2,3,4)$, or X contains a proper closed orbit Z which is one of the following

(c1) Z is a flag manifold, hence a homogeneous divisor;

(c2) Z is the projective space $\mathbf{P}^{n-1}$ and the blow-up of X along Z is a quasihomogeneous variety with the exceptional divisor being homogeneous;

(c3) Z is the Grassmann manifold of codimension 2.

Remark . Except in the case (c3), we end up with an orbit which is a divisor. Therefore either the complement of the open orbit is disconnected or it consists of this divisor and we can apply the results of Akhiezer. If $n = 3$, however, the cases (c2) and (c3) in the above theorem coincide so we get the result by [Na2] (and [An]).
For case (c3) extra work is needed.
Note that the next case, namely $dimX = 2n-1$ is a very interesting one, corresponding to the Mu-Um case of 3-folds and $SL(2)$-action.

Sketch of proof: an application of B-L-V theorem.

Let $G/P = Z \subset X$ be a closed positive dimensional orbit of the G action with normal bundle $N_{Z/X}$; note that $N_{Z/X}$ is homogeneous with respect to the tangent action of G.
It is natural to believe that the action of G on $N_{Z/X}$ should somehow reflect the action of G on X. A naive expectation is that if the action of G on X has an open orbit then the induced action of G on $N_{Z/X}$ has an open orbit as well (as in the case of fixed point). However it turns out that, in general, the expectation is wrong, as showed by:

Example . Let us consider the action of $G = SL(2)$ on $X = \mathbf{P}^3 = \mathbf{P}(\Lambda^3\mathbf{C})$. The only closed orbit of the action is a rational twisted cubic $Z \subset \mathbf{P}^3$ and there is moreover an open orbit and a unique invariant 2-dimensional subset $Y \supset Z$ which is swept by lines tangent to Z. We blow up $\mathbf{P}^3$ along Z and then the resulting 3-fold along the intersection of the exceptional divisor with the strict transform of Y, $B = \widehat{Z} \cap \widehat{Y}$, which is the unique closed orbit.
Call the result $\widetilde{X}$ with the exceptional divisor $\widetilde{B}$ and $\widetilde{Z}, \widetilde{Y}$ denoting strict transforms of the respective divisors. We lift up the action of $SL(2)$ to $\widetilde{X}$. The above three named divisors are clearly $SL(2)$ invariant and they intersect transversally along a 1-dimensional orbit C of the action – transversality of the intersection can be verified along a 2-dimensional slice of $\widetilde{X}$ over a curve in $\mathbf{P}^2$.
Thus the induced action of $SL(2)$ on $\mathbf{P}(N_{C/\widetilde{X}})$ has three non-meeting orbits at least, which is possible only if $\mathbf{P}(N_{C/\widetilde{X}}) = \mathbf{P}^1 \times \mathbf{P}^1$ and the action is nontrivial along one coordinate only.

Set-up and Notation. Let $G/P = Z \subset X$ be a closed positive dimensional orbit of the G action; let $\widetilde{P} \subset G$ be a subgroup of G opposite to P with $\widetilde{P}^u$ denoting its radical unipotent part and $L = P \cap \widetilde{P}$ its Levi subgroup, so that $\widetilde{P} = \widetilde{P}^u \cdot L$.

BLV Theorem . [Brion-Luna-Vust, Thm 1.4] *There exists a locally closed affine subvariety $W \subset X$ such that*

(1) $z \in W$ and W in L-invariant,

(2) $\widetilde{P}^u \cdot W$ is an open subset of X,

(3) the action of $\widetilde{P}^u$ on X induces an isomorphism of algebraic varieties $\widetilde{P}^u \times W \simeq \widetilde{P}^u \cdot W$.

Corollary. *In the above notation suppose that a general orbit of the action of $\widetilde{P}$ on X is of dimension k. Then a general orbit of the normal representation $L \to GL((N_{Z/X})_z)$ is of dimension $\geq k - dimZ$.*

Proof of the Corollary. Let O be a general orbit of the action of $\widetilde{P}$ on the open subset $X_0 = \widetilde{P}^u \cdot W$ which, by assumption, has dimension k; We note that, because $\widetilde{P} = \widetilde{P}^u \cdot L$ and W is L invariant, we have $\widetilde{P} \cdot (O \cap W) = \widetilde{P}^u \cdot (O \cap W) \simeq \widetilde{P}^u \times (O \cap W)$ where the latter isomorphism follows by property (3) of BLV Theorem. Thus $dim(O \cap W) = dimO - dim\widetilde{P}^u = dimO - dimZ$. On the other hand (again by (3) and $\widetilde{P} = \widetilde{P}^u \cdot L$), the stabilizer subgroup of the set W in $\widetilde{P}$ is L, hence $O \cap W$ is an orbit for the action of L on W.
Since L is reductive and W is a smooth affine L-variety with a fixed point z, we can apply Local Linearization Theorem in order to compare the action of L around $z \in W$ with its tangent action $L \to GL(T_z W)$. In particular this last action has an orbit of dimension $k - dimZ$. But the representation $L \to GL(T_z W)$ is just the restriction to L of the normal representation $P \to GL((N_{Z/X})_z)$.

In order to apply the corollary we need informations on the action of $\widetilde{P}$ on X as in the following:

Theorem. *Let X be a $SL(n)$ quasihomogeneous manifold which contains a closed orbit $Z \simeq \mathbf{P}^{n-1}$. Let $P = G_z \subset SL(n)$ be the isotropy group of a point $z \in Z$ and let $\widetilde{P} \subset SL(n)$ be a parabolic subgroup opposite to P. If $dimX \leq 2n-2$ then $\widetilde{P}$ acts on X with an open orbit.*

The classification theorem follows then by the following steps
0- By the theorem of Luna the tangent representation at a fixed point has an open orbit, therefore there are no fixed point if $n < dimX < 2n$. We can thus assume that there exists an orbit $\simeq \mathbf{P}^n$.
1- The last theorem plus the corollary of BLV imply that the normal representation as an open orbit.
2- By the **classification of homogeneous vector bundle on $\mathbf{P}^{n-1}$**, we have then that if $dimX \leq 2n-2$ one of the following occurs:

(a) $dimX = n$ and $N_{Z/X} = \mathcal{O}(a)$ with $a \neq 0$, or

(b) $dimX = 2n-2$ and either $N_{Z/X} \simeq T\mathbf{P}^{n-1}(a)$ or $N_{Z/X} \simeq \Omega\mathbf{P}^{n-1}(a)$ with $a \in \mathbf{Z}$.

It remains to give a **proof of the theorem**

Lemma. *Let $H, H' \subset G$ be connected closed algebraic subgroups of G with the associated Lie algebra $\mathcal{H}, \mathcal{H}' \subset \mathcal{G}$. Then the left action of H' on G/H has an open orbit if and only if for some $g \in G$ we have $\mathcal{H}' + Ad(g)(\mathcal{H}) = \mathcal{G}$. Note that the above statement is symmetric (that is the action of H' on G/H has an open orbit if and only if the action of H on G/H' has an open orbit).*

Thus we have to prove the following: if $W \subset sl(n)$ is a linear subspace of codimension $\leq 2n-2$ and $\mathcal{P} \subset sl(n)$ be a maximal parabolic subalgebra of codimension $n-1$ then for a general $g \in SL(n)$ we have $Ad(g)(\mathcal{P}) + W = sl(n)$.
We can identify $sl(n)$ with the space of sections of $T(G/P) \simeq T\mathbf{P}^{n-1}$ and we let $\epsilon : \mathbf{P}^{n-1} \times sl(n) \to T\mathbf{P}^{n-1}$ be the evaluation map. Our claim follows if the restriction $\epsilon_W : \mathbf{P}^{n-1} \times W \to T\mathbf{P}^{n-1}$ is generically surjective.
This is the case since otherwise we let $\mathcal{F} := im(\epsilon_W)$ and we have a contradiction from

Lemma. *Let $\mathcal{F}$ be a coherent $\mathcal{O}$-subsheaf of $T\mathbf{P}^{n-1}$ of rank $< n-1$. Then $dimH^0(\mathcal{F}) \leq n^2 - 2n$.*

Remark . The bound $dimH^0(\mathcal{F}) \leq n^2 - 2n$ of the lemma is optimal as one can see taking for instance $n = 3$, a point $x \in \mathbf{P}^2$ and the exact sequence

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow T\mathbf{P}^2 \longrightarrow \mathcal{J}_x(2) \longrightarrow 0.$$

Problem Does a similar lemma hold for other closed orbit $Z = G/P$, i.e. for the Grassamanians?

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Marco Andreatta
Dipartimento di Matematica, Universitá di Trento, 38050 Povo (TN), Italy
e-mail : andreatt@science.unitn.it