Extending extremal rays and contractions from an ample section

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Notations

- X is a smooth complex projective variety of dimension n
- $\mathcal{E}$ is an ample vector bundle of rank r on X
- $Z = (s = 0) \subset X$ with $s \in \Gamma(\mathcal{E})$ is a smooth submanifold with $\dim Z = \dim X - r = n - r$ (an ample section).

Problem

Ascend the geometric properties of Z to get informations on the geometry of X.

(A very classical problem in projective geometry, see Castelnuovo, Enriques,...., Badescu, Fujita, Sommese,....)

In the Mori theory spirit:

Assume that Z is not minimal, i.e. Z has at least one extremal ray in the negative part of the Mori cone; does this ray (or the associated extremal contraction) determine a ray (or a contraction) in X, and if so, does this new ray determine the structure of X?

General principles

- A variety is as special as any of its ample sections
- Some variety Z are so special that they cannot be sections of any variety X.

Simple features of our set up

Normal bundle:

$$N_{Z/X} = \mathcal{E}_Z$$

and therefore

$$(K_X + det\mathcal{E})_Z = K_Z$$

Weak Lefschetz:

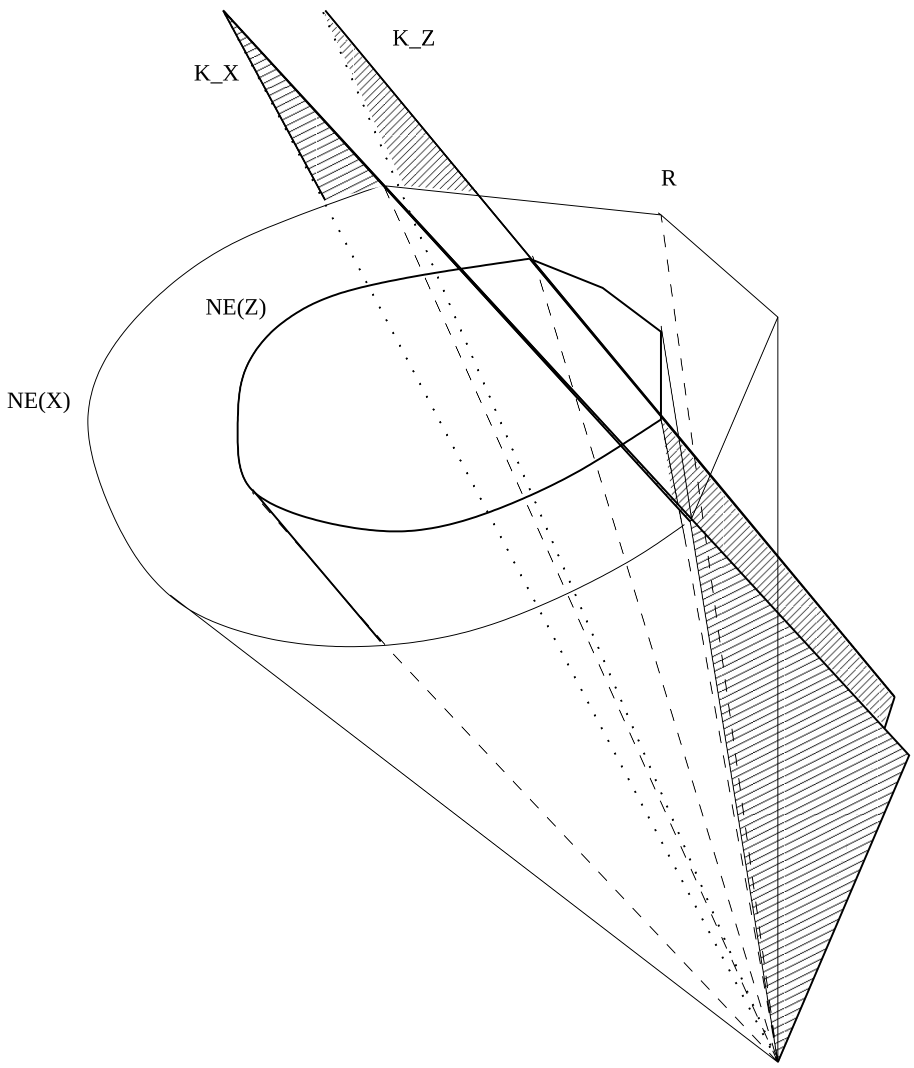
$$i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$$

is an isomorphism if $\dim Z \geq 3$
and is injective if $\dim Z = 2$.

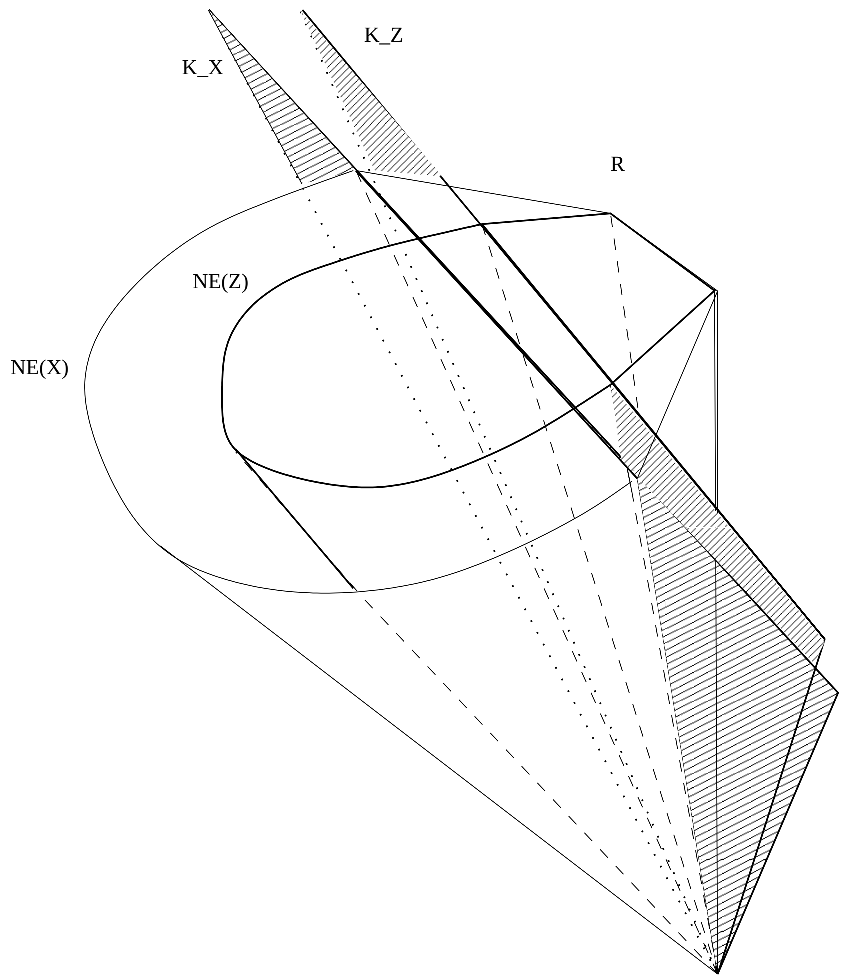
Ampleness:

$$Y \subset X \text{ with } \dim Y \geq r$$
$$\Downarrow$$
$$\dim Z \cap Y \geq \dim Y - r$$

Thus X has an extremal ray if Z has; if i^* is an isomorphism $N_1(X) = N_1(Z)$ and the Kleiman-Mori cones $NE(Z) \subset NE(X)$ look like:



Theorem. *The following is the actual picture*

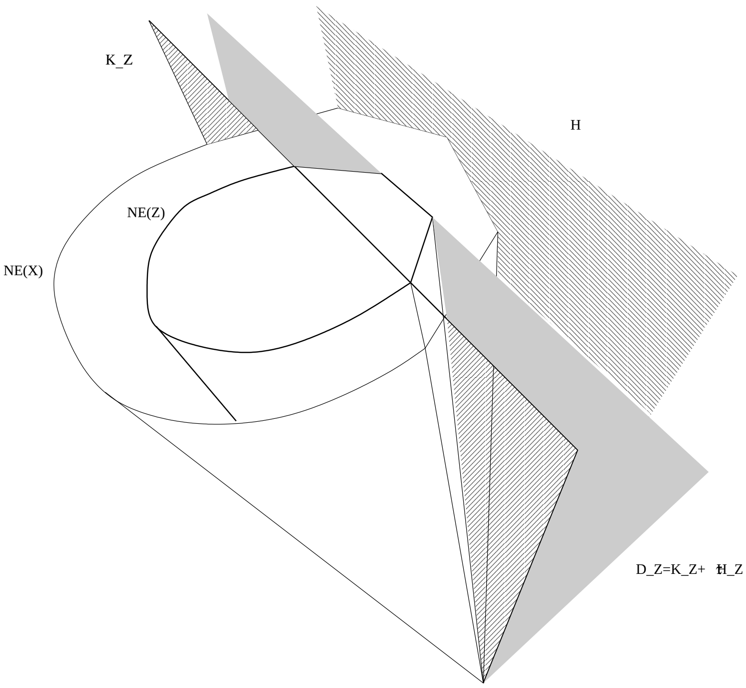


That is there exists at least a ray in Z wich is also a ray in X (we will call it a liftable ray). If the two cones have part of a face in common then the entire face is common to both cones.

Scketch of proof

Let H ample on X.
$\exists \tau \in \mathbb{Q}$ s.t. $D_Z = K_Z + \tau H_Z$ is nef, not ample and it defines a contraction $\varphi : Z \to W$.

Claim: $D = K_X + det\mathcal{E} + \tau H$ is nef, not ample and it defines a contraction $\phi : X \to Y$.



If by contradiction $D.C < 0$ then there existe an extremal ray such that $D.R < 0$. Consider

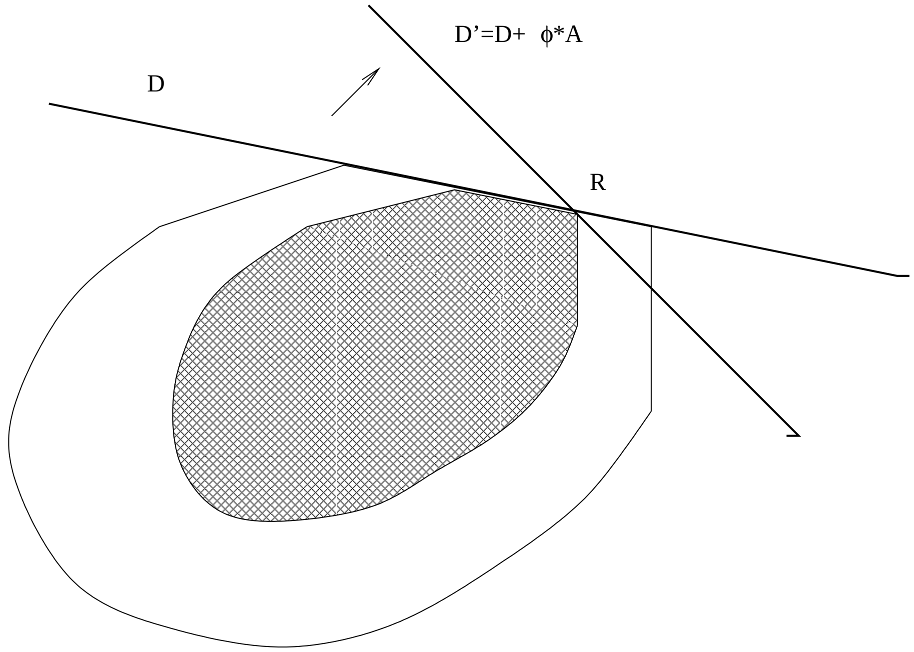
the contraction associated to R , $\rho : X \to W$, and let F a component of a fiber.
Since $dim(F \cap Z) \geq dimF - r$, we have that $dimF \leq r$, since D_Z is nef.
On the other hand

$$dimF \geq \min\{-K_X C : C \in R\} \geq r$$

with equality only if ρ is of fiber type (Ionescu-Wisniewski, from bend and break).

Thus $\dim F = r$ for all F, X is a $\mathbb{P}^r$-bundle su W, and Z a regular section; $Z \simeq W$ is a contradiction with Weak Lefschetz.

We move D and do the same for every ray in the face


D'=D+ ϕ*A
D
R

Let R be a common ray and let ϕ and φ be the associated contraction on X and on Z respectively. We have the commutative diagram

$$\begin{tikzcd} X & Z \arrow[l, "i"'] \arrow[d, "\varphi"] \arrow[dl, "\phi_Z"'] \\ Y \arrow[from=u, "\phi"'] & W \arrow[l, "\pi"] \end{tikzcd}$$

where $\pi : W \to \phi_Z(Z)$ is a finite morphism. If it is an isom. then we say that R is extendable

- φ is of fiber $\Rightarrow$ π isom. and ϕ is of fiber type.
- φ is a $\mathbb{P}-$bundle $\Rightarrow$ ϕ is also a $\mathbb{P}-$bundle.

- φ is birational then ϕ may be not birational; if not then ϕ is a scroll and π is an isomorphism.
-φ is a blow-up of a smooth subvariety of codim $\geq 3 \Rightarrow \phi$ is also a smooth blow-up.

Example $X = \mathbb{P}^2 \times \mathbb{P}^1$, $Z \simeq \mathbb{F}_1 = Bl(\mathbb{P}^2) \in |\mathcal{O}(1,1)|$.

Examples

Let W be a smooth projective variety and let

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0$$

be a vector bundle exact sequence.

$$\phi_Z : Z = \mathbb{P}(\mathcal{F}) \to W \subset \phi : X = \mathbb{P}(\mathcal{G}) \to W$$

is a section of $\mathcal{E} = \xi_{\mathcal{G}} \otimes \phi^* \mathcal{I}\check{}$ corresponding to

$$\phi^* \mathcal{I} \to \phi^* \mathcal{G} \to \xi_{\mathcal{G}}.$$

If $\mathcal{E} = \xi_{\mathcal{G}} \otimes \phi^* \mathcal{I}\check{}$ is ample we have examples; note that this implies $dimW < (g - i + 1)$ where $i = rk\mathcal{I}$ and $g = rk\mathcal{G}$, otherwise there is a degenerate locus of the above map.

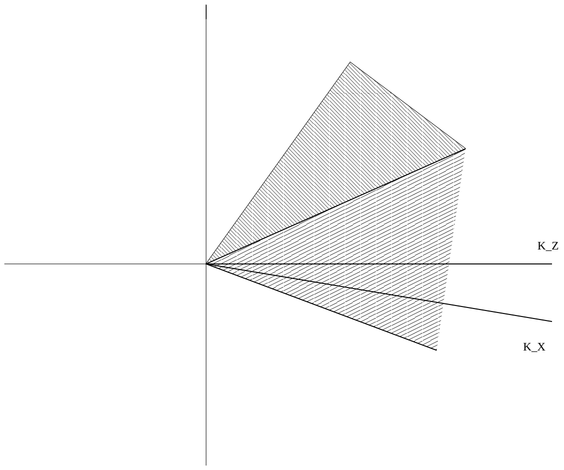
For instance

$$0 \to \oplus^n \mathcal{O}_{\mathbb{P}^1} \to \oplus^n(\mathcal{O}_{\mathbb{P}}(a) \oplus \mathcal{O}_{\mathbb{P}}(s-a)) \to \oplus^n \mathcal{O}_{\mathbb{P}}(s) \to 0$$

with a, s such that $0 < s - a < a$.

This gives $Z := \mathbb{P}^1 \times \mathbb{P}^{n-1}$ as a (ample) section of $\mathcal{E} = \oplus^n \xi_{\mathcal{G}}$ on $X = \mathbb{P}(\mathcal{G})$, where $\mathcal{G} = \oplus^n(\mathcal{O}_{\mathbb{P}}(a) \oplus \mathcal{O}_{\mathbb{P}}(s-a))$.

The cones are described in the following picture



In particular there is a ray in common (the one associated to the curve contracted by the projections to $\mathbb{P}^1$). In general X is not Fano and therefore the other rays are different (i.e. the contraction to the second factor cannot be lifted).

Sommese's conjecture *Z is a $\mathbb{P}^k$-bundle over $W \Rightarrow X$ is a $\mathbb{P}^{k+r}$-bundle over W and $k \geq dimW - 1$ with two exceptions: the above example and $Z = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ or in $\mathbb{Q}^3$.*

Theorem. *Sommese's conjecture is true if W is minimal.*

The theorem follows from the above cone theorem once we notice that Z has only one extremal ray (for this we use an argument of Szurek-Wiśniewski and of Kollár-Miyaoka-Mori) and from a characterization of scrolls via adjunction theory.

Theorem. *Sommese's conjecture is true if $W = S$ is a surface with $k(S) \geq 0$ (and thus for all surface by previus results).*

The proof is based on a "relative" M. M. P. together with the study of rays on Z and on the theorem of extensibility of blow-ups.

For blow-ups we have similar theorems, as said above, in particular

Remark: Fujita proved that smooth blow-up are liftable (and extendable) from an ample divisor.

Theorem. *If* $Z = Bl_{\mathbb{P}^{k-m}}(\mathbb{P}^k)$ *with* $m \geq 3$*, then* $X = \mathbb{P}(\mathcal{G}) \longrightarrow \mathbb{P}^{m-1}$ *for* $\mathcal{G}$ *v. b. on* $\mathbb{P}^{m-1}$ *with*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow \oplus^{k-m-1}\mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(1) \longrightarrow 0$$

and $\mathcal{E} = \xi_{\mathcal{G}} \otimes g^*\mathcal{I}^{\smile}$.

Corollary. $Z = Bl_{\mathbb{P}^{k-m}}(\mathbb{P}^k)$ *with* $m \geq 3$ *cannot be ample section of a bundle which is a direct sum of line bundles. There are examples for non splitting vector bundle.*

A main problem

Can one say a priori which are the (or some of the) liftable extremal faces on Z?

Conjecture. *If R is an extremal ray on Z with length bigger then r then it is liftable* (from a similar conjecture of Sommese).

And also: do there exist liftable but not extaendable contractions?