Milan J. Math. Vol. 77 (2009) 361–383
DOI 10.1007/s00032-009-0097-4
Published online August 8, 2009

Milan Journal of Mathematics

A Conjecture of Mukai Relating Numerical Invariants of Fano Manifolds

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Abstract. A complex manifold X of dimension n such that the anti-canonical bundle $-K_X := \det \mathrm{TX}$ is ample is called a Fano manifold. Besides the dimension, other two integers play an essential role in the classification of these manifolds, namely the pseudoindex of X, i_X, which is the minimal anticanonical degree of rational curves on X, and the Picard number ρ_X, the dimension of $N_1(X)$, the vector space generated by irreducible complex curves modulo numerical equivalence . A (generalization of a) conjecture of Mukai says that $\rho_X(i_X - 1) \leq n$. In this paper we present some partial steps towards the conjecture, we show how one can interpretate and possibly solve it with the use of families of rational curves on a uniruled variety, and more generally with the instruments of Mori theory. We consider also other related problems: the description of some Fano manifolds which are at the border of the Mukai relations and how the pseudoindex changes via (some) birational transformation.

Mathematics Subject Classification (2000). Primary 14J45; Secondary 14E30.

Keywords. Fano manifolds, rational curves, Picard number, pseudoindex.

1. Introduction

Let X be a smooth complex variety of dimension n and let TX be its tangent bundle. If $n = 1$ a beautiful theorem, which goes back to Riemann and Poincaré, says that we can find a hermitian metric on TX whose curvature is constant; moreover if the curvature is positive then X is $\mathbb{P}^1$, if it is zero

then X is a torus, while if it is negative then X is a Riemann surface of genus ≥ 2.
Passing to higher dimensions the situation becomes much more intricated. Even the concept of positivity for vector bundles has many different possible extensions: a vector bundle $\mathcal{E}$ on a smooth complex variety X which has a hermitian metric with positive curvature tensor is said to be Griffiths–Nakano positive (or ample). But one can define ampleness in the algebraic enviroment (Grothendieck) and it turns out that these definitions are equivalent only for vector bundles of rank one, not for higher rank.
In accordance with conjectures of Frenkel and Hartshorne, Mori, in 1979 [Mor79], proved that $\mathbb{P}^n$ is the only smooth complex variety with ample tangent bundle (whatever condition of ampleness you assume, that is algebraic or complex analytic).
In higher dimensions one can define positivity of a complex manifold in different ways: for instance one can simply assume that a locally free subsheaf $\mathcal{E} \subset \mathrm{TX}$ is ample. Mori's proof extends also in this apparently more general set up and it was proved that this is the case if and only if $X = \mathbb{P}^n$ ([AW01]; see also [Ara06]).

Some really new manifolds appear if we simply assume that the determinant of the tangent bundle, the anticanonical bundle $-K_X := \det \mathrm{TX}$, is ample. Manifolds with ample anticanonical bundle are called Fano manifolds. These manifolds are a natural generalization of the Riemann sphere $\mathbb{P}^1$ in higher dimensions; this class obviously contains $\mathbb{P}^n$ (the determinant of an ample vector bundle is ample) and more generally all smooth hypersurfaces of $\mathbb{P}^{n+1}$ of degree less than $n+1$.
A deep result of Mori, fundamental for the above mentioned paper as well in many subsequent ones, says that Fano manifolds are covered by rational curves, that is by curves which are birational to $\mathbb{P}^1$. In this sense Fano manifolds belong to a larger class of projective manifolds which are called uniruled, i.e. manifolds covered by rational curves.

Another weak condition on positivity of TX, which as well generalizes the one dimensional condition, is the following: TX is said to be not generically seminegative if there exist a subsheaf $\mathcal{E} \hookrightarrow \mathrm{TX}$ and a family of curves $\{C_t\}$ covering an open subset of X such that $\mathcal{E}_{C_t}$ is locally free and $\deg(\mathcal{E})_{C_t} > 0$. With a technical theorem Miyaoka proved that TX is not generically seminegative if and only if X is uniruled [Miy87].
Uniruled manifolds seem to be the correct generalization of $\mathbb{P}^1$ in higher dimensions from the point of view of classification of complex projective

manifolds. For this however we have to introduce mild types of singularities which we will not discuss in this paper.

The following facts are equivalent:

a) X is uniruled;
b) $\kappa(X) = -\infty$ (κ is the Kodaira dimension);
c) X is birational to a fibration in (possibly singular) Fano varieties;
d) TX is not generically seminegative.

The equivalence between a) and d) is the theorem of Miyaoka; c) implies a) is a theorem of Miyaoka and Mori which generalizes the theorem of Mori on existence of rational curves on Fano manifolds; a) implies b) is straightforward. Finally b) implies c) is a conseguence of the existence of minimal models in the Minimal Model Program (MMP), which has been proved by Mori [Mor88] in dimension 3 and which has been recently proved in any dimension by Birkar, Cascini, Hacon and McKernan [BCHM06].

As a consequence Fano manifolds become the building blocks for a classification of uniruled manifolds. Let us mention that Fano manifolds are also very important in complex differential geometry. In particular compact complex manifolds with a positive Kähler–Einstein metric are Fano. On the other hand it is a very hard problem to check whether a Fano manifold has a Kähler–Einstein metric or not (see [DK01]).

All above says that it is important in complex geometry to understand and classify Fano manifolds. This is not hopeless since it has been proved that in any given dimension there are finitely many deformations type of Fano manifolds.
Fano manifolds of dimension two are called del Pezzo surfaces and they were classified by classical algebraic geometers. Fano manifolds of dimension three were first studied by Gino Fano, subsequently by Iskovskikh and finally by Mori and Mukai, who, with a last remark in 2002, obtained a complete classification. In higher dimensions the classification depends on a bunch of numerical invariants and on the relations between them.

The main topic of this exposition is to discuss a conjecture of Mukai which states a relation between the dimension, the pseudoindex and the Picard number of a Fano manifold. In particular we present an attempt of proof, developed in the paper [ACO04], which establishes some special cases; in section 4 we will give a slight improvement of the results in that paper. We consider also other related problems, as for instance the description of some Fano manifolds which are at the border of the Mukai relations and the study of how the pseudoindex changes by birational transformation.

2. Mukai conjecture

Let X be a Fano manifold of dimension n, that is a complex manifold whose anticanonical bundle, $-K_X := \det \mathrm{TX}$ is ample.

We denote by $N_1(X)$ the vector space generated by irreducible complex curves modulo numerical equivalence and by ρ_X its dimension, which is called Picard number; we also denote by $\mathrm{NE}(X)$ the cone inside $N_1(X)$ spanned by the classes of effective curves.

Definition 2.1. The index of a Fano manifold X is the integer defined as

$$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some Cartier divisor } L\},$$

while the pseudoindex is defined as

$$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \subset X \text{ rational curve }\}.$$

Remark 2.2. The pseudoindex is an integral multiple of the index: if $-K_X = r_X L$ and $i_X = -K_X \cdot C'$ then $i_X = ar_X$, where $a = L \cdot C'$.
If $\rho_X > 1$ the notion of index seems to be less useful, as the following example shows: let $X = \mathbb{P}^n \times \mathbb{P}^{n+1}$; we have $r_X = 1$ and $i_X = n + 1$.

Remark 2.3. The definition of pseudoindex requires the existence of rational curves on a Fano manifold. This was proved by Mori who also proved that $i_X \leq (n+1)$. A classical result of Kobayashi–Ochiai [KO72] says that $r_X \leq n+1$ with equality if and only if $X = \mathbb{P}^n$. The much stronger result that $i_X = n+1$ if and only if $X = \mathbb{P}^n$ is also true and it was proved recently by Cho, Miyaoka, Sheperd-Barron [CMSB02] and Kebekus [Keb02].

In 1988 Mukai [Muk88] stated the following conjecture:

Conjecture 2.1 (Mukai conjecture). *Let X be a Fano manifold and let n, ρ_X, r_X be respectively its dimension, its Picard number and its index. Then*

$$\rho_X(r_X - 1) \leq n;$$

moreover equality holds if and only if $X \simeq (\mathbb{P}^{r_X-1})^{\rho_X}$.

If $\rho_X = 1$ the conjecture follows from the Kobayashi–Ochiai result; in this case moreover the much stronger result with the pseudoindex is true (Mori). Since for $\rho_X > 1$ the pseudoindex seems to be more appropriate the following generalization of the conjecture was proposed in [BCDD03]:

Conjecture 2.2 (Generalized Mukai conjecture (GM-conjecture)). *Let X be a Fano manifold and let n, ρ_X, i_X be respectively its dimension, its Picard number and its pseudoindex. Then*

$$\rho_X(i_X - 1) \leq n;$$

moreover equality holds if and only if $X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}$.

Let us briefly recall the existing results towards the conjectures.

Fano manifolds of dimension $n \leq 3$ are classified and for them GM-conjecture holds.

In 1990 Wiśniewski [Wiś90] proved that if $i_X > \frac{n+2}{2}$ then $\rho_X = 1$; moreover if $r_X = \frac{n+2}{2}$ then either $\rho_X = 1$ or $X = (\mathbb{P}^{r-1})^2$. The case $i_X = \frac{n+2}{2}$ was settled by Occhetta in [Occ06].

In 2001 Cho, Miyaoka and Shepherd Barron [CMSB02] proved that if $i_X \geq n+1$ then $X \simeq \mathbb{P}^n$. All together these prove GM-conjecture for $i_X \geq \frac{n+2}{2}$.

In 2002 Bonavero, Casagrande, Debarre and Druel [BCDD03] proved that GM-conjecture holds if one of the following is true:

a) $n = 4$; b) X is homogeneous; c) X is toric and $i_X \geq \frac{n+3}{3}$ or $n \leq 7$.

In 2004 Andreatta, Chierici and Occhetta [ACO04] proved that GM-conjecture holds if one of the following is true:

a) $n = 5$; b) $i_X \geq \frac{n+3}{3}$ and there exists an unsplit dominating family of rational curves for X; c) $i_X \geq \frac{n+3}{3}$ and either X has a contraction of fiber type or X does not have small contractions.

The definition of unsplit dominating family of rational curve is a technical one which will be presented in the next section. Fano manifolds with no unsplit dominating families of rational curves do exists (see example 3 and theorem 5.1), even if we don't know an example with $i_X \geq \frac{n+3}{3}$.

More generally one can prove that GM-conjecture holds if $i_X \geq \frac{n+k}{k}$ and there exists $(k-2)$ families of rational curves which are unsplit and dominating.

Finally in 2006 Casagrande [Cas06] proved that GM-conjecture holds for toric varieties.

In the following two sections we will present some technical tools and the main ideas which are beyond the proofs of the above quoted special cases (except for the toric case, which has a completely different approach, typical of that category).

3. Rational curves on Fano manifolds

A rational curve on a (normal) projective variety X is, by definition, a morphisms $f : \mathbb{P}^1 \to X$ which is birational onto its image. Results concerning the existence of rational curves and the study of schemes parametrizing them are the starting points of Mori theory. To use them we have to set up some technical definitions and to recall some basic results; our notation is consistent with the one in [Kol96] to which we constantly refer.

Let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the scheme parametrizing morphisms $f : \mathbb{P}^1 \to X$. We consider the open subscheme $\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$, corresponding to those morphisms which are birational onto their image, and its normalization $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$. The group $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$, the quotient exists and it will be denoted by $\mathrm{Ratcurves}^n(X)$. The quotient of the product action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1$ will be denoted by $\mathrm{Univ}(X)$.

Definition 3.1. A family of rational curves is an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$.

Given a family $V \subseteq \mathrm{Ratcurves}^n(X)$, we denote by V_x the subscheme of V parametrizing rational curves passing through x.

Given a rational curve $f : \mathbb{P}^1 \to X$ we will call a family of deformations of f any irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ containing the equivalence class of f.

We define a Chow family of rational curves to be an irreducible component $\mathcal{V} \subset \mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles. If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$ is called the Chow family associated to V.

If $L \in \mathrm{Pic}(X)$ is a line bundle, we will denote by $L \cdot V$ the intersection number of L and a general member of the family V; moreover we will denote by $[V]$ the numerical equivalence class in $N_1(X)$ of a general member of the family V.

Given a family V of rational curves, we have the following basic diagram:

$$\begin{array}{ccc} p^{-1}(V) =: U & \xrightarrow{i} & X \\ {\scriptstyle p}\downarrow & & \\ V & & \end{array} \tag{3.1}$$

where i is the map induced by the evaluation $ev : \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X$ and p is the $\mathbb{P}^1$-bundle induced by the projection $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$. We define $\mathrm{Locus}(V)$ to be the image of U in X and $\mathrm{Locus}(V_x)$ to be the image of $p^{-1}(V_x)$.

Some families of rational curves are special and their existence determine the geometry of X.

Definition 3.2. Let V be a family of rational curves on X. Then

(a) V is unsplit if it is proper;
(b) V is locally unsplit if for the general $x \in \mathrm{Locus}(V)$ every component of V_x is proper;
(c) V is generically unsplit if through two general points, which are on a curve in V, there is at most a finite number of curves of V.
(d) V is quasi-unsplit if every component of any reducible cycle in $\mathcal{V}$ is numerically proportional to V.

Note that $(a) \Rightarrow (b) \Rightarrow (c)$; the first implication is clear, while to prove the second Mori's bend and break lemmata are needed (see [Kol96], IV.2.3). It is also obvious that $(a) \Rightarrow (d)$.

Example. Let $\sigma : X \to \mathbb{P}^5$ be the blow up of $\mathbb{P}^5$ along a two-dimensional quadric Q, let Λ be the three dimensional linear subspace which contains Q and let E be the exceptional divisor; the family V of deformations of the strict tranform of a line of $\mathbb{P}^5$ which does not meet Q is generically unsplit, but not locally unsplit: given $x, y \in X$ such that the line l through $\sigma(x), \sigma(y)$ does not meet Q the only curve of V passing through x and y is the strict transform of the line l, but, for every $x \in \mathrm{Locus}(V)$ the curves of V_x degenerate in a cycle with two components: the strict tranform of a line $\bar{l}$ through $\sigma(x)$ which meets Q in a point p and a line in the fiber of σ over p.

In this example moreover, the family W of deformations of the strict tranform of a line of $\mathbb{P}^5$ which meets Q in a point is locally unsplit, but not unsplit. In fact for a general point $x \in X$ curves in W_x do not degenerate, but if x is a point in E the curves of W_x degenerate in a cycle with two components: the strict tranform of a line $\bar{l}$ through $\sigma(x) \in Q$, which meets Q in another point p, and a line in the fiber of σ over p.

Condition (d) is independent from (b) and (c) : the families V and W of the above example are respectively generically unsplit and locally unsplit, but none of them is quasi-unsplit.

On the other hand the family of conics in $\mathbb{P}^2$ is quasi unsplit, but it is not generically unsplit (hence not locally unsplit and not unsplit).

Definition 3.3. Let U be an open dense subset of X and $\pi : U \to Z$ a proper surjective morphism to a quasi projective variety; we say that V is a dominating family with respect to π if the locus of the curves in V not contracted by π dominates Z; if $Z = X$ we call V a dominating family. If X admits dominating families with respect to π, we can choose among these families one with minimal degree with respect to a fixed ample line bundle, and we call it a minimal horizontal dominating family with respect to π; if $Z = X$ we will call it a minimal dominating family.

It is not difficult to prove that minimal horizontal dominating families are locally unsplit.

The geometry of Fano manifolds is strongly related to the properties of families of rational curves of low degree. The first result in this direction is a fundamental theorem, due to Mori:

Theorem 3.4. [Mor79] *Through every point of a Fano manifold X there exists a rational curve C such that $-K_X \cdot C \leq \dim X + 1$.*

A relative version of Mori's theorem, [KMM92, Theorem 2.1], states that, if X is a Fano manifold, U is an open dense subset of X and $\pi : U \to Z$ is a proper surjective morphism on a quasi projective variety Z of positive dimension, for a general point $z \in Z$, there exists a rational curve C on X of anticanonical degree $\leq \dim X + 1$ which meets $\pi^{-1}(z)$ without being contained in it (an horizontal curve, for short).

Remark 3.5. As a consequence, we have that on Fano manifolds there exist dominating families and horizontal dominating families (with respect to suitable morphisms π) of rational curves; this is because there are only a finite number of families $\{V^i \subset \mathrm{Ratcurves}^n(X)\}$ of degree $\leq \dim X + 1$. Taking minimal families with this properties we find locally unsplit dominating (or horizontal dominating) families. However, we can not expect to find always unsplit dominating families, as example 3 above shows.

Definition 3.6. Let $Y \subset X$ be a closed subset and $\mathcal{V}^1, \ldots, \mathcal{V}^k$ be Chow families of rational curves.

We define $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ to be the set of points $x \in X$ such that there exist cycles $C_1, \ldots, C_k$ with the following properties: C_i belongs to the family $\mathcal{V}^i$, $C_i \cap C_{i+1} \neq \emptyset$, $C_1 \cap Y \neq \emptyset$ and $x \in C_k$. I.e. $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$

is the set of points that can be joined to Y by a connected chain of k cycles belonging respectively to the families $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

We define $\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ to be the set of points $x \in X$ such that there exist cycles $C_1, \ldots, C_m$ with the following properties: C_i belongs to a family $\mathcal{V}^j$, $C_i \cap C_{i+1} \neq \emptyset$, $C_1 \cap Y \neq \emptyset$ and $x \in C_m$. I.e. $\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ is the set of points that can be joined to Y by a connected chain of at most m cycles belonging to the families $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

Remark 3.7. Note that

$$\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y = \bigcup_{1 \leq i(j) \leq k} \mathrm{Locus}(\mathcal{V}^{i(1)}, \ldots, \mathcal{V}^{i(m)})_Y.$$

We define a relation of rational connectedness with respect to $\mathcal{V}^1, \ldots, \mathcal{V}^k$ on X in the following way: x and y are in $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$ relation if there exists a chain of cycles in $\mathcal{V}^1, \ldots, \mathcal{V}^k$ which joins x and y, i.e. if $y \in \mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_x$ for some m.
The $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$ relation just defined is, in the language of [Kol96, IV.4.8], nothing but the set theoretic relation $\langle \mathcal{U}_1, \ldots, \mathcal{U}_k \rangle$ associated to the proper proalgebraic relation $\mathrm{Chain}(\mathcal{U}_1, \ldots, \mathcal{U}_k)$.

We have the following:

Theorem 3.8. [Cam92] *or* [Kol96, IV.4.16]. *There exist an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to Z^0$ such that the rc$(\mathcal{V}^1, \ldots, \mathcal{V}^k)$ relation restricts to an equivalence relation on X^0, the fibers of π are equivalence classes for the rc$(\mathcal{V}^1, \ldots, \mathcal{V}^k)$ relation and any two points in a fiber can be connected by a chain of at most $2^{\dim X - \dim Z^0} - 1$ cycles in $\mathcal{V}^1, \ldots, \mathcal{V}^k$.*

Definition 3.9. The map π is called the $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$ fibration. We will say that X is rationally connected with respect to $(\mathcal{V}^1, \ldots, \mathcal{V}^k)$ if $\dim Z_0 = 0$.

Remark 3.10. If V is an unsplit family, then V corresponds to the normalization of the associated Chow family $\mathcal{V}$; in particular V itself defines a proper prerelation, so, to our purposes we can identify V with $\mathcal{V}$.

Proposition 3.11. [ACO04, Corollary 4.2] *Every curve in* $\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ *is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles parametrized by $\mathcal{V}^1, \ldots, \mathcal{V}^k$.*

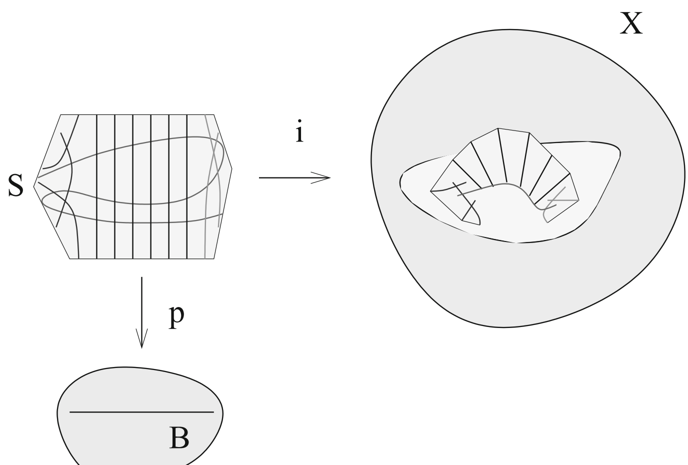
This follows from the above Remark 3.10 together with the following fundamental non-breaking lemma, ([ACO04, Lemma 4.1]), of which we will give the proof.

Lemma 3.12. *Let $Y \subset X$ be a closed subset, $\mathcal{V}$ a Chow family of rational curves. Then every curve contained in* $\mathrm{Locus}(\mathcal{V})_Y$ *is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and of irreducible components of cycles parametrized by $\mathcal{V}$ which meets Y.*

Proof. Consider the universal family over $\mathcal{V}$:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{i} & X \\ \downarrow{\scriptstyle p} & & \\ \mathcal{V} & & \end{array}$$

Let $\mathcal{V}_Y = p(i^{-1}(Y \cap \mathrm{Locus}(\mathcal{V})))$, and let $\mathcal{U}_Y = p^{-1}(V_Y)$. Let C be a curve in $\mathrm{Locus}(\mathcal{V})_Y$ which is not an irreducible component of a cycle parametrized by $\mathcal{V}$. Then $i^{-1}(C)$ contains an irreducible curve C' which is not contained in a fiber of p and dominates C via i. Let $B = p(C')$ and let S be the surface $p^{-1}(B)$.
Note that there is a curve C'_Y in S which dominates B and such that $i(C'_Y)$ is contained in Y: this is due to the fact that the image via i of every fiber of $p_{|S}$ meets Y.



By [Kol96, II.4.19] every curve in S is algebraically equivalent to a linear combination with rational coefficients of C'_Y and of the irreducible components of fibers of $p_{|S}$ (in [Kol96, II.4.19] take $X = S$, $Y = B$ and $Z = C'_Y$). Thus any curve in $i(S)$, in particular C, is algebraically, hence numerically, equivalent in $i(\mathcal{U}_Y) = \mathrm{Locus}(\mathcal{V})_Y$ (and hence in X) to a linear combination with rational coefficients of $i_*(C_Y)$ and of irreducible components of cycles parametrized by $\mathcal{V}_Y$. $\square$

Unfortunately, in general there is no way of counting how many numerically independent irreducible components can have the cycles parametrized by a fixed Chow family, unless it is (quasi-)unsplit; actually this is the reason that motivated the introduction of quasi-unsplit families in [CO06a]. From Proposition 3.11 the next Corollary follows almost straightforward.

Corollary 3.13. [ACO04, Corollary 4.4] *If X is rationally connected w.r.t. some quasi-unsplit families $V^1, \ldots, V^k$, then $N_1(X) = \langle [V^1], \ldots, [V^k] \rangle$; in particular $\rho_X \leq k$.*

The following is another basic tool for the present approaches to Mukai's conjecture.

Proposition 3.14. [Kol96, IV.2.6] *Let X be a smooth projective variety, V a family of rational curves and $x \in \mathrm{Locus}(V)$ a point such that every component of V_x is proper. Then*

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq \dim X - K_X \cdot V - 1.$$

The same result holds for a general $x \in \mathrm{Locus}(V)$ assuming that V is generically unsplit.

Lemma 3.15. [ACO04, Cfr. Lemma 5.4] *Let $Y \subset X$ be a closed subset and V an unsplit family. Assume that $[V] \notin NE(Y)$ and that $Y \cap \mathrm{Locus}(V) \neq \emptyset$. Then for a general $y \in Y \cap \mathrm{Locus}(V)$*

(a) $\dim \mathrm{Locus}(V)_Y \geq \dim(Y \cap \mathrm{Locus}(V)) + \dim \mathrm{Locus}(V_y)$;
(b) $\dim \mathrm{Locus}(V)_Y \geq \dim Y - K_X \cdot V - 1$.

Moreover, if $V^1, \ldots, V^k$ are numerically independent unsplit families such that curves contained in Y are numerically independent from curves in $V^1, \ldots, V^k$ then either $\mathrm{Locus}(V^1, \ldots, V^k)_Y = \emptyset$ or

(c) $\dim \mathrm{Locus}(V^1, \ldots, V^k)_Y \geq \dim Y + \sum -K_X \cdot V^i - k$.

Proof. We refer to the diagram 3.1. Since V is unsplit, for a point y in $Y \cap \mathrm{Locus}(V)$ we have

$$\dim(i^{-1}(y)) = \dim V_y = \dim \mathrm{Locus}(V_y) - 1.$$

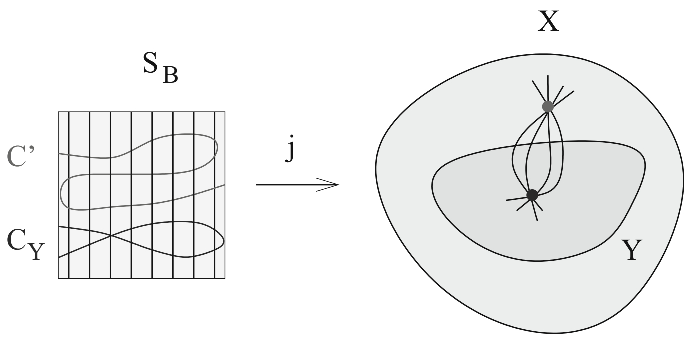
So, setting $V_Y = p(i^{-1}(Y))$ and $U_Y = p^{-1}(V_Y)$, we have for general $y \in Y \cap \mathrm{Locus}(V)$,

$$\begin{aligned}\dim U_Y &= \dim(Y \cap \mathrm{Locus}(V)) + \dim \mathrm{Locus}(V_y) \geq \\ &\geq \dim Y + \dim \mathrm{Locus}(V) - n + \dim \mathrm{Locus}(V_y) \geq \\ &\geq \dim Y + \deg V - 1.\end{aligned}$$

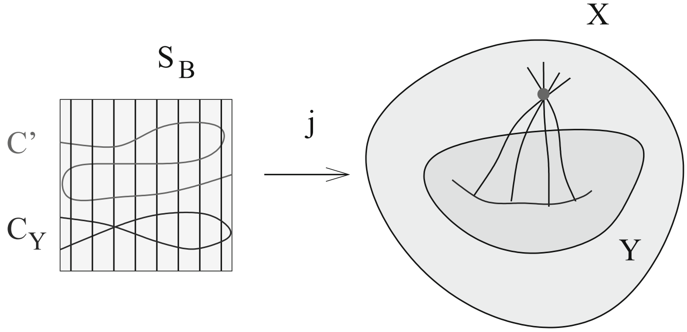
Since $\mathrm{Locus}(V)_Y = i(U_Y)$, (a) and (b) will follow if we prove that $i : U_Y \to X$ is generically finite.
To show this we take a point $x \in i(U_Y)\backslash Y$ and we suppose that $i^{-1}(x) \cap U_Y$ contains a curve C' which is not contained in any fiber of p; let B' be the curve $p(C') \subset V_Y$ and let $\nu : B \to B'$ be the normalization of B'.
By base change we obtain the following diagram

$$\begin{array}{ccc} S_B & \xrightarrow{j} & X \\ \downarrow{\scriptstyle p_B} & & \\ B & & \end{array}$$

Let C_Y be a curve in S_B which dominates B and whose image via j is contained in Y; such a curve exists since the image via j of every fiber of p_B meets Y. Now two cases are possible: either $j(C_Y)$ is a point, and therefore we have a one-parameter family of curves passing through two fixed points, contradicting the fact that V is unsplit (and therefore generically unsplit):



or $j(C_Y)$ is a curve in $Y \cap \mathrm{Locus}(V_y)$, so a curve in Y is numerically proportional to a curve parametrized by V, against the assumptions:



To show (c) it is enough to recall that

$$\mathrm{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y = \mathrm{Locus}(\mathcal{V}^k)_{\mathrm{Locus}(\mathcal{V}^1,\dots,\mathcal{V}^{k-1})_Y}. \qquad \square$$

Corollary 3.16. [BCDD03] *If on a Fano manifold X of pseudoindex i_X and Picard number ρ_X there exist $V^1, \dots, V^{\rho_X}$ unsplit families of rational curves whose numerical classes are numerically independent and such that* $\mathrm{Locus}(V^1, \dots, V^{\rho_X})_x \neq \emptyset$ *for some $x \in X$, then $\rho_X(i_X - 1) \leq \dim X$.*

Proof. By lemma 3.15 (c) we have

$$\dim X \geq \dim \mathrm{Locus}(V^1, \dots, V^{\rho_X})_x \geq \rho_X(-K_X \cdot V^i) - \rho_X \geq \rho_X(i_X - 1).$$

$\square$

3.1. Extremal rays: basic definitions

Let X be a projective manifold of dimension n. Let also $\overline{\mathrm{NE}(X)} \subset N_1(X)$ be the closure of the cone spanned by effective curves in X, the so called Mori–Kleiman cone. The famous "Cone Theorem" of Mori says that the part of this cone over which K_X is negative is generated by countably many *extremal rays* and that these rays can only accumulate at the hyperplane $K_X = 0$. Moreover each extremal ray is spanned by a rational curve, i.e. $R = \mathbb{R}_+[C]$ with C a rational curve such that $0 < -K_X \cdot C \leq n + 1$. In particular if X is Fano $\mathrm{NE}(X)$ is a closed polyhedral cone generated by a finite number of extremal rays.

Another basic step of the Mininimal Model Program is the "Contraction Theorem" (by Kawamata and Shokurov), which essentially says that given an extremal ray R there exists a unique morphism $\varphi_R : X \longrightarrow Z$ onto a normal projective variety Z, with connected fibers such that a curve $C \subset X$ is contracted by φ_R to a point in Z if and only if $[C] \in R$. φ_R is

called the (elementary) contraction associated to R.
The exceptional locus of φ is given by

$$\mathrm{Exc}(R) = \{x \in X : x \in C \ \text{ a rational curve in } R\}.$$

The ray R, equivalently the contraction φ_R, is said to be of fiber type, resp. of birational type, resp. of small type if $\dim \mathrm{Exc}(R)$ is equal to n, resp. less than n, resp. less than $n-1$.

4. Proofs

In this section we will show how to prove GM-conjecture for a Fano manifold X of dimension n and pseudoindex $i_X \geq \frac{n+3}{3}$ with an unsplit dominating family of rational curves, which is the easiest case considered in [ACO04]; we will also prove a new result (Theorem 4.3) which gives some more evidence towards the general case.

Theorem 4.1. *Let X be a Fano manifold of dimension n and pseudoindex $i_X \geq \frac{n+3}{3}$; if there exists a family V of rational curves which is unsplit and covering then Conjecture GM is true for X.*

Proof. Consider the rc$V$ fibration $\pi : X^0 \to Z^0$: if $\dim Z^0 = 0$ then $\rho_X = 1$ by corollary 3.13 and GM-conjecture holds. Otherwise take a minimal horizontal dominating family V'; such a family is locally unsplit and a general curve of V' is numerically independent from curves contracted by π; picking a general $x \in \mathrm{Locus}(V')$ and applying lemma 3.15 (b) with $Y = \mathrm{Locus}(V'_x)$ we obtain

$$\begin{aligned} n &\geq \dim \mathrm{Locus}(V)_{\mathrm{Locus}(V'_x)} \geq \\ &\geq \dim \mathrm{Locus}(V'_x) - K_X \cdot V - 1 \geq \\ &\geq -K_X \cdot V' - K_X \cdot V - 2 \end{aligned}$$

so $K_X \cdot V' \leq 2i_X - 1$ and therefore V' is actually an unsplit family.
Let $\pi' : X' \to Z'$ be the rc(V, V') fibration: if $\dim Z' = 0$ then from corollary 3.13 we have $\rho_X = 2$ and we conclude. Otherwise we take a minimal horizontal dominating family with respect to π', V''.
For general $x \in \mathrm{Locus}(V'')$ denote by F the fiber of π' containing x: then F is an equivalence class with respect to the rc(V, V') relation, so, if $y \in F$ then $F \supseteq \mathrm{Locus}(V, V')_y$. By lemma 3.15 (c)

$$\dim F \geq -K_X \cdot V - K_X \cdot V' - 2 \geq 2i_X - 2.$$

Since a general curve of V'' is numerically independent from curves contracted by π' we have $\dim(\mathrm{Locus}(V''_x) \cap F) = 0$, so

$$n \geq \dim F + \dim \mathrm{Locus}(V''_x) \geq 2i_X - 2 - K_X \cdot V'' - 1,$$

that is

$$-K_X \cdot V'' \leq n + 3 - 2i_X \leq i_X.$$

This is impossible unless $-K_X \cdot V = -K_X \cdot V' = -K_X \cdot V'' = i_X$ and $\dim \mathrm{Locus}(V_x) = \dim \mathrm{Locus}(V'_x) = \dim \mathrm{Locus}(V''_x) = i_X - 1$. Proposition 3.14 implies that all these families are covering, so we can apply [Occ06, Theorem 1] to obtain that $X \simeq (\mathbb{P}^{i_X-1})^3$. □

We would like to propose here a slight improvement of the results in [ACO04].

Proposition 4.2. *Let X be a Fano manifold of dimension n and pseudoindex $i_X \geq \frac{n+3}{3}$; let V be a minimal dominating family of rational curves, and let $\mathcal{V}$ be the associated Chow family. If V is not unsplit then X is rc$\mathcal{V}$-connected.*

In particular the numerical class of V does not lie in a proper face of $NE(X)$.

Proof. Let $X \stackrel{\pi}{\dashrightarrow} Z$ be the rc$\mathcal{V}$ fibration; since V is not unsplit we have $-K_X \cdot V \geq 2i_X$, hence, for a general $x \in X$, we have $\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V - 1 \geq 2i_X - 1$. The rc$\mathcal{V}$ fibration contracts $\mathrm{Locus}(V_x)$ to a point, and therefore $\dim Z \leq \dim X - (2i_X - 1) < i_X - 1$.
Suppose by contradiction that $\dim Z > 0$ and let V' be a minimal horizontal dominating family with respect to π. Let x be a general point in $\mathrm{Locus}(V')$; we have $\dim \mathrm{Locus}(V'_x) \geq i_X - 1$ and $\dim \mathrm{Locus}(V'_x) \leq \dim Z$, since curves of V' are not contracted by π; therefore $\dim Z \geq i_X - 1$, a contradiction.

The last sentence follows from the fact that, since X is rc$\mathcal{V}$ connected, $N_1(X)$ is generated by irreducible components of cycles in $\mathcal{V}$. □

Theorem 4.3. *Let X be a Fano manifold of dimension n and pseudoindex $i_X \geq \frac{n+3}{3}$; if GM-conjecture is not true for X then for every proper face σ of $NE(X)$ any two extremal rays, R_1 and R_2, in σ will have non intersecting exceptional loci.*

Proof. Let V be a minimal dominating family for X; since we are assuming that GM-conjecture does not hold then V is not unsplit by theorem 4.1; therefore X is rc$\mathcal{V}$ connected by proposition 4.2.

If V is quasi unsplit then $\rho_X = 1$, so we can assume that V is not quasi-unsplit.
Assume that there exists a proper face σ of $\mathrm{NE}\,(X)$ containing two extremal rays R_1 and R_2 with meeting exceptional loci and let V^1 and V^2 be minimal degree families of rational curves in R_1 and R_2 such that $\mathrm{Locus}(V^1) \cap \mathrm{Locus}(V^2) \neq \emptyset$.
By lemma 3.15, for $y \in \mathrm{Locus}(V^1) \cap \mathrm{Locus}(V^2)$, we have

$$\dim \mathrm{Locus}(V^1, V^2)_y \geq -K_X \cdot (V^1 + V^2) - 2 \geq 2i_X - 2.$$

Let $x \in X$ be a general point; since X is rc$\mathcal{V}$-connected we can join x and y with a chain of cycles belonging to $\mathcal{V}$; denote by Γ the first cycle in this chain meeting $\mathrm{Locus}(V^1, V^2)_y$.
We can assume that Γ is reducible; this follows from the fact that, for any point z in $\Gamma \cap \mathrm{Locus}(V^1, V^2)_y$ we have that V_z is not unsplit.
In fact, if this were not the case, every curve in $\mathrm{Locus}(V_z)$ would be numerically proportional to V; moreover, by the usual dimension estimates $\dim(\mathrm{Locus}(V_z) \cap \mathrm{Locus}(V^1, V^2)_y) > 0$ but every curve in $\mathrm{Locus}(V^1, V^2)_y$ has numerical class contained in the face σ, hence the numerical class of V would be contained in σ, against Proposition 4.2.
Therefore we have proved that there exists a component Γ' of a reducible cycle $\Gamma = \Gamma' + \Gamma''$ in $\mathcal{V}$ meeting $\mathrm{Locus}(V^1, V^2)_y$ without being contained in it; let W' be a family of deformations of Γ' and let W'' be a family of deformations of Γ''.
Note that either $[W']$ or $[W'']$ are not contained in σ, since, by corollary 4.2, $[V] = [W' + W'']$ is not contained in σ.
If the numerical class of W' is not contained in σ then, by lemma 3.15 we have $\dim \mathrm{Locus}(V^1, V^2, W')_y \geq \dim X$, and therefore W' is covering, against the minimality of V.
If the numerical class of W' is contained in σ then take i such that W' is independent from V^i; by the same lemma we have $\dim \mathrm{Locus}(V^i, W', W'')_y \geq \dim X$, and therefore W'' is covering, again contradicting the minimality of V. $\square$

Remark 4.4. We do not know any Fano manifold (no assumptions on the pseudoindex, Picard number clearly greater or equal than three) for which there does not exists a proper face σ in $\mathrm{NE}\,(X)$ containing two extremal rays R_1 and R_2 with meeting exceptional loci.

In [ACO04] GM-conjecture was proved to be true also for fivefolds; the proof is quite long and involved, and it uses ad hoc arguments, essentially based on the construction of special divisors which are irreducible components of loci of unsplit families.

5. Related problems

In this section we will survey some topics related to GM-conjecture.

5.1. Classification of some Fano fivefolds with $\rho_X \geq 1$ and $r_X \geq 2$

The partial proof of GM-conjecture gave, in many cases, more informations on the cones of curves. Building upon these partial results, a complete and effective classification of the possible cones of curves of Fano manifolds of pseudoindex and Picard number greater or equal than two has been achieved in [CO06a, Theorem 1.1]; it means that the possibilities for the number and the types of extremal rays of the cone are given. Quite surprisingly, all these cones are simplicial, i.e. the number of extremal rays equals the Picard number; moreover in almost all cases the manifolds admit a fiber type extremal ray. In the remaining cases X does not admit a quasi-unsplit minimal dominating family of rational curves and a complete classification is given:

Theorem 5.1. [CO06a, Theorem 1.2] *Let X be a Fano fivefold of pseudoindex two which does not have a quasi-unsplit locally unsplit dominating family of rational curves; then $\rho_X = 2$ and X is the blow-up of $\mathbb{P}^5$ along a two-dimensional smooth quadric or along a cubic scroll in $\mathbb{P}^4$, or along a Veronese surface.*

Other results for Fano fivefolds of index two can be found in [NO07] or [CO06b].

5.2. Extremality of quasi-unsplit families

The results in [CO06a] shows that, for a Fano fivefold of index greater than one, the existence of a quasi-unsplit covering family V is equivalent to the existence of a fiber type contraction of a ray spanned by the class of V.
It is easy to show that, if a manifold X has an elementary fiber type contraction, then it admits a covering quasi-unsplit family of rational curves (just take a minimal dominating family of curves in the ray). To the author's knowledge there are no examples in which the contrary is not true,

that is in which there exists a covering quasi-unsplit families whose numerical class is not extremal.
Some results in this direction have been obtained in the recent paper [BCD07], in which the authors prove that, if an n-dimensional variety X admits a covering quasi unsplit family of rational curves V such that the dimension of a general rc$\mathcal{V}$ equivalence class is greater or equal than $n-3$, then the numerical class of this family has to be extremal.

5.3. Mukai conjecture restricted to a ray

Given an extremal ray R of a smooth variety X of dimension n, we can define the length of R as

$$l(R) := \min\{-K_X \cdot C \mid C \text{ a rational curve in } R\};$$

the length is thus a sort of pseudoindex restricted to the curves in a ray. In particular the length is greater or equal to the pseudoindex and, by the Cone Theorem, it is less or equal than $n+1$, equality holding if and only if $X \simeq \mathbb{P}^n$ (again by the results in [CMSB02] or [Keb02]). The length of an extremal ray of birational type is bounded above by $n-1$, equality holding if and only if the associated contraction is the blow-up of a point in a smooth variety (see [AO02, Theorem 1.1]).

The following theorem can be thought as a sort of "weighted Mukai conjecture", taking into account the length of a ray R; it is actually an improvement of GM-conjecture in the case $\rho_X = 2$. We denote by $Bl_Y(X)$ the blow up of X along Y.

Theorem 5.2. [AO05, Theorems 1.1 and 1.2] *Let X be a Fano manifold of pseudoindex i_X and $\rho_X > 1$; let R be an extremal ray of X, of length $l(R)$ and with exceptional locus* $\mathrm{Exc}(R)$. *Then*

$$i_X + l(R) \leq \dim \mathrm{Exc}(R) + 2.$$

If equality holds and R is not small then $X \simeq \mathbb{P}^k \times \mathbb{P}^{n-k}$ or $X \simeq Bl_{\mathbb{P}^t}(\mathbb{P}^n)$ with $0 \leq t \leq \frac{n-3}{2}$.
If equality holds for the index, namely $r_X + l(R) = \dim \mathrm{Exc}(R) + 2$, then, $X = \mathbb{P}_{\mathbb{P}^k}(\mathcal{O}^{\oplus e-k+1} \oplus \mathcal{O}(1)^{\oplus n-e})$, with $k = n-r+1$ and $e = dim(\mathrm{Exc}(R))$.

We also considered the next step, namely

$$i_X + l(R) = \dim \mathrm{Exc}(R) + 1.$$

In this case, for a fiber type or divisorial extremal ray R, we proved that $\rho_X \leq 3$, we described the Kleiman–Mori cone of X and we classified the varieties with $\rho_X = 3$. Moreover, assuming that R is the ray associated to a smooth blow-up, we gave a complete classification:

Theorem 5.3. [AO05, Theorem 1.3] *Let X be a Fano manifold and let R an extremal ray whose associate contraction $\varphi_R : X \to Y$ is the blow up of a smooth variety Y along a smooth subvariety $T \subset Y$, such that*

$$i_X + l(R) \geq n \qquad \text{or equivalently} \qquad i_X \geq \dim T + 1.$$

Then X is one of the following

a) $Bl_{\mathbb{P}^t}(\mathbb{P}^n)$, *with $\mathbb{P}^t$ a linear subspace of dimension $\leq \frac{n}{2} - 1$,*
b) $Bl_{\mathbb{P}^t}(\mathbb{Q}^n)$, *with $\mathbb{P}^t$ a linear subspace of dimension $\leq \frac{n}{2} - 1$,*
c) $Bl_{\mathbb{Q}^t}(\mathbb{Q}^n)$, *with $\mathbb{Q}^t$ a smooth quadric of dimension $\leq \frac{n}{2} - 1$ not contained in a linear subspace of $\mathbb{Q}^n$,*
d) $Bl_p(V)$ *where V is $Bl_Y(\mathbb{P}^n)$ and Y is a submanifold of dimension $n-2$ and degree $\leq n$ contained in an hyperplane H such that $p \notin H$,*
e) $Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$.

Note that if T is a point the condition $i_X \geq \dim T + 1 = 1$ is empty. In this case the theorem is actually the main theorem of [BCW02], where Fano manifolds which are the blow-up at a point of a smooth variety are classified (those varieties correspond to cases a), b) with $t = 0$ and d) of the above theorem).

5.4. Relate the pseudoindex of (some) birational Fano manifolds

The results in theorem 5.3 show that if a Fano manifold X is the blow-up of a manifold Y along a manifold T and $i_X \geq \dim T + 1$ then also Y is a Fano manifold and $i_Y \geq i_X$.
In general this is not true; in [Wiś91, Section 3] the question whether Y has to be a Fano variety was posed and some answers were given in [Wiś91, Propositions 3.4 and 3.6].
In particular the examples [Wiś91, 3.7, 3.8] show that $i_T \geq \dim T + 1$ is the best possible bound which guarantees that Y is a Fano manifold.

Assuming that Y is Fano one can ask a second problem, namely can the pseudoindex of Y be less than the pseudoindex of X? This has been studied in [Bon05], where the following example was given.

Example. Let $Y_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^m}^{\oplus m} \oplus \mathcal{O}_{\mathbb{P}^m}(1))$ and let $T_n \subset Y_n$ be the submanifold defined by the subbundle $\mathcal{O}^{\oplus m}$. Note that $\dim Y_n := n = 2m$ and $\dim T_n = m$. Let $\varphi_n : X_n = Bl_{T_n}(Y_n) \longrightarrow Y_n$ be the blow-up of Y_n along T_n. If $n \geq 4$

one can easily prove that X_n and Y_n are Fano manifolds, that $i_{Y_n} = 1$, $i_{X_4} = 1$ and $i_{X_n} = 2$ for $n \geq 6$.

The following are the main results of [Bon05]:

Theorem 5.4. *Let X be a Fano manifold of dimension n which is the blow up $\varphi_R : X \longrightarrow Y$ of a smooth Fano manifold Y along a smooth subvariety T.*

- *If $2\dim T \leq n+i_Y-1$ then $i_X \leq i_Y$ unless $n \geq 6$ is even and $X = X_n$, $Y = Y_n$, $T = T_n$ are as in the above example.*
- *If $i_Y \geq \frac{n}{3}-1$ then $i_X \leq i_Y$ unless $n = 6$ and $X = X_6$, $Y = Y_6$, $T = T_6$ are as in the above example.*

We propose here a slight variation of the results of [Bon05], considering birational contractions between smooth Fano manifolds:

Proposition 5.5. *Let X be a Fano manifold, let $\varphi_R : X \longrightarrow Y$ be the contraction of a birational extremal ray R such that Y is a smooth Fano manifold and let $T = \varphi_R(\mathrm{Exc}(R))$. If $i_Y > 2\dim T + 1 - n$ or if $i_Y > \frac{n}{3} - 1$ then $i_X \leq i_Y$.*

Proof. Since φ is a birational map between smooth varieties the exceptional locus $\mathrm{Exc}(R)$ is a divisor and we have the canonical bundle formula:

$$K_X = \varphi_R^* K_Y + G,$$

where G is a divisor supported on $\mathrm{Exc}(R)$.

Let $C \subset Y$ be a rational curve such that $i_Y = -K_Y \cdot C$ and let V be a family of rational curves on Y containing C.

By 3.14 we have $2\dim \mathrm{Locus}(V) \geq n+i_Y-1$, therefore if $i_Y > 2\dim T + 1-n$ we have that $\dim \mathrm{Locus}(V) > \dim T$ and this implies that there exists a curve C in V not contained in T.
The strict transform of it, call it $\widetilde{C}$, is a rational curve on X satisfying $G \cdot \widetilde{C} \geq 0$, therefore, by the canonical bundle formula, $i_X \leq -K_X \cdot \widetilde{C} = -K_Y \cdot C - G \cdot \widetilde{C} \leq i_Y$.

Assume now that $i_Y > \frac{n}{3} - 1$ and, by contradiction, that $i_X > i_Y$; by the first part we can assume that $i_Y \leq 2\dim T + 1 - n$.
Denote by F a general fiber of the map φ; from 3.14 we have $\dim F \geq i_X$ and therefore

$$i_Y \leq 2\dim T + 1 - n = n - 1 - 2\dim F \leq n - 1 - 2i_X \leq n - 3 - 2i_Y$$

that is $i_Y \leq \frac{n}{3} - 1$, which is a contradiction. □

Corollary 5.6. *Let X be a Fano manifold, let $\varphi_R : X \longrightarrow Y$ be the contraction of a birational extremal ray R such that Y is a smooth Fano manifold and let $T = \varphi_R(\mathrm{Exc}(R))$. If $i_X > \dim T - i_Y$ (for instance if $i_X \geq \dim T$) then $i_X \leq i_Y$.*

Proof. Let F be a general fiber of φ_R; we have

$$\dim T \leq \dim \mathrm{Exc}(R) - \dim F \leq n - 1 - i_X \leq n - \dim T + i_Y - 1$$

so that

$$2 \dim T - n + 1 < i_Y.$$

We can thus apply proposition 5.5 to conclude. □

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Received: February 2009.