Minimal Model Program and Adjunction Theory

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Quasi polarized pairs

Let X be a projective variety with terminal and $\mathbb{Q}$-factorial singularities of dimension n.
Let L be a Cartier divisor (a line bundle) which is ample, or simply nef and big.

The pair (X, L) is called a polarized pair, or a quasi polarized pair.

For instance let $X \subset \mathbb{P}^N$ be a projective variety and $L := \mathcal{O}(1)_{|X}$, or better its (partial) desingularizaton and the pull back of L.

Classical problems

Problem Given a general element $D \in |L|$
(assume that X is not a cone over D).

Which properties of D lift to X;
do these properties determine X ?

Enriques–Castelnuovo studied the case in which X is a surface and D is a curve of low genus, or of minimal degree, ...
Sakai studied the case in which X is a normal surface.
Fano studied the case in which X is a 3-fold and D is a $K3$ surface.
Mori in his first paper proved that if D is a c.i. in a weighted projective space the same is for X.
Sommese proved that abelian and bi-elliptic surfaces **cannot be ample sections**, unless X is a cone.

Adjunction Theory

Adjunction Theory wants to classify quasi polarized pairs via the study of the nefness of the adjont bundles

$$K_X + rL,$$

with r natural (or rational) positive number.

Assume that there exist r sections of $|L|$ which intersect in a $n-r$ variety D, with terminal singularities ($r = n-1$, we ask for a smooth curve).

To get nefness of $K_X + rL$ implies, by adjunction $(K_X + rL)_{|D} = K_D$, to get a minimal model for D.

Minimal Model Program- BCHM

A log pair (X, Δ), i.e a normal variety X and an effective $\mathbb{R}$ divisor Δ, is Kawamata log terminal (klt) if:
$K_X + \Delta$ is $\mathbb{R}$-Cartier and for a (any) log resolution $g : Y \to X$ we have $g^*(K_X + \Delta) = K_Y + \Sigma b_i \Gamma_i$ with $b_i < 1$, for all i.

By BCHM on a klt log pair (X, Δ), with Δ big, we can run a

$K_X + \Delta$- Minimal Model Program with scaling:

$$(X_0, \Delta_0) = (X, \Delta) \to (X_1, \Delta_1) \to - - -- \to (X_s, \Delta_s)$$

Minimal Model Program- BCHM

$$(X_0, \Delta_0) = (X, \Delta) \to (X_1, \Delta_1) \to - - -- \to (X_s, \Delta_s)$$
such that:

1) (X_i, Δ_i) is a klt log pair, for $i = 0, ..., s$;

2) $\varphi_i : X_i \to X_{i+1}$ is a birational map which is either a divisorial contraction or a flip associated with an extremal ray $R_i = \mathbb{R}^+[C_i]$ such that $(K_{X_i} + \Delta_i) \cdot C_i < 0$
(notation: $R_i \in \overline{NE(X_i)}_{(K_{X_i}+\Delta_i)<0} \subset \overline{NE(X_i)}_{K_{X_i}<0}$)

3) either $K_{X_s} + \Delta_s$ is nef (i.e. (X_s, Δ_s) is a log Minimal Model),
or $X_s \to Z$ is a Mori fiber space relatively to $K_{X_s} + \Delta_s$
(depending on the pseudeffectivity of $K_X + \Delta$).

MMP for a q.p. pair

Let (X, L) be a quasi-polarized variety: let $r \in \mathbb{Q}^+$.

Lemma (zip L into a boundary). Since L is nef and big there exists an effective $\mathbb{Q}$-divisor Δ^r on X such that

$$rL \sim_{\mathbb{Q}} \Delta^r \quad \text{and} \quad (X, \Delta^r) \text{ is Kawamata log terminal.}$$

Run a $K_X + \Delta^r$-MMP and get a birational klt pair (X_s, Δ^r_s) which is
- either a Minimal Model ($K_{X_s} + \Delta_s$ is nef)
- or $X_s \to Z$ is a Mori fiber space relatively to $K_{X_s} + \Delta_s$.

Remarks/Problems

- (X_s, Δ^r_s) is not necessarily an (r) q.p. pair, i.e. we do not have a priori a nef and big Cartier divisor L_s such that $rL_s \sim_{\mathbb{Q}} \Delta^r_s$.
- Beyond the existence of the MMP, it would be nice to have a "description" of each steps and eventually of the Mori fiber spaces.

Extremal rays

For the above program we study the (Fano-Mori) contractions:

$$\varphi : X \to Y$$

associated to a rays $R = \mathbb{R}^+[C] \in \overline{NE(X)}_{(K_X+rL)<0} \subset \overline{NE(X)}_{K_X<0}$.

That is φ is a projective map between normal variety, with connected fibers, X has terminal $\mathbb{Q}$-factorial singularities and an irreducible curve $C \subset X$ is mapped to a point by φ iff $[C] \in R$.

- φ can be of fiber type ($dimX > dimY$), φ is called a Mori fiber space
- or birational, φ can then be divisorial or small

Let F be a non trivial fiber of φ; we possibly restrict to an affine neighborhood of the image of F (local set up).
Then $(K_X + \tau L) \sim_\varphi \mathcal{O}_X$ for a rational $\tau > r$.

Apollonio method

Let $X' \in |L|$ a generic divisor with "good singularities". We have:

- $\varphi_{|X'} := \varphi' : X' \to Y'$ is a contraction with connected fibre, around $F' := F \cap X'$;
it is the Fano-Mori contraction associate to $R' \in \overline{NE(X')}_{(K_{X'}+(r-1)L')<0}$.

- Any section of L on X' extends to a section of L on X.

Base point free technique

Theorem [Fano, Fujita, Kawamata, Kollar, ...,
...., A-Wisniewski, Mella, A-Tasin]

- $dimF > (r-1)$; if φ is birational then $dimF > r$.
- If $\dim F < r+1$, or $\dim F \leq r+1$ if φ is birational, then L is very ample (relatively to φ).
- If $\dim F < r+2$ then there exists $X' \in |L|$ with "good" singularities (i.e. as in X). The same is true if $\dim F = r+2$, except for two cases in which $n = 3$ and φ is of fiber type.

Weighted Blow-up

Let $X = \mathbb{A}^n = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$, $Z = \{x_1 = \dots = x_k = 0\} \subset X$ and $\mathbb{P}(a_1, \dots, a_k)$ the weighted projective space with weight $(a_1, \dots, a_k)$.

Consider the rational map

$$\varphi : \mathbb{A}^n \to \mathbb{P}(a_1, \dots, a_k)$$

given by $(x_1, \dots, x_n) \mapsto (x_1^{a_1} : \dots : x_k^{a_k})$.

The *weighted blow-up* of X along Z with weight $\sigma = (a_1, \dots, a_k, 0, \dots, 0)$ is defined as the closure $\overline{X}$ in $\mathbb{A}^n \times \mathbb{P}(a_1, \dots, a_k)$ of the graph of φ, together with the morphism $\pi : \overline{X} \to X$ given by the projection on the first factor.
The map π is birational and contracts an exceptional irreducible divisor E to Z. Moreover for any point $z \in Z$ we have $\pi^{-1}(z) = \mathbb{P}(a_1, \dots, a_k)$.

Weighted Projective Space

For any $d \in \mathbb{N}$ we define the σ-weighted ideal of degree d as

$$I_{\sigma,d} = \{g \in \mathbb{C}[x_1, \dots, x_n] : \sigma\text{-wt}(g) \geq d\} = (x_1^{s_1} \cdots x_n^{s_n} : \sum_{j=1}^{k} s_j a_j \geq d).$$

The weighted blow-up of $X = \mathbb{A}^n$ along $Z = \{x_1 = \ldots = x_k = 0\}$ with weight $\sigma = (a_1, \dots, a_k, 0, \dots, 0)$, $\pi : \overline{X} \to X$, is given by

$$\overline{X} = \operatorname{Proj} \bigoplus_{d \geq 0} I_{\sigma,d} \to X.$$

Castelnuovo-Kawakita

Theorem

Let $\varphi : X \to Y$ be a birational contraction associated with an extremal ray $R = \mathbb{R}^+[C]$ on a q.p. pair, such that $L \cdot C > 0$ and $R = \mathbb{R}^+[C] \in \overline{NE(X)}_{(K_X+(n-2)L)<0} \subset \overline{NE(X)}_{K_X<0}$.
(i.e.a birational map in a $K_X + \Delta^{n-2}$-MMP)

Then $\varphi : X \to Y$ is the weighted blow-up of a smooth point in Y of weights $(1, 1, b, ..., b)$, where b is a natural positive number.

$L' = \varphi_*(L)$ is a Cartier divisor on Y such that $\varphi^* L' = L + bE$, where E is the exceptional (Weil) divisor.

Definition

We call such φ a Castelnuovo-Kawakita contraction.

Proof

We have $\dim F > (n-2)$; thus $\dim F = (n-1)$ and φ is a contraction of a divisor to a point.
By the base point free theorem, we can assume L is very ample.

Thus we get the existence of sections in $|L|$ with terminal singularities. Inductively, slicing with $(n-2)$ general sections of $|L|$, we can reduce to the case of a Fano Mori contraction on a surface.
Surfaces with terminal singularities are smooth. Apply now Castelnuovo's Theorem to have that the image is smooth .

Since Y has terminal $\mathbb{Q}$-factorial singularities this implies that Y is smooth at the exceptional point.

Proof

Since $X = \mathrm{Proj}_{\mathcal{O}_Z}(\oplus_{d\geq 0} f_*(\mathcal{O}_X(-dbE))$, we want to prove that

$$f_*(\mathcal{O}_X(-dbE) = (x_1^{s_1}\cdots x_n^{s_n} \mid s_1 + s_2 + \sum_{j=3}^{n} bs_j \geq db).$$

a) consider a general element $X' \in |L|$; the restricted morphism $f' := f_{|X'} : X' \to Y' := f(X')$ is a divisorial Fano-Mori contraction of a ray $R' \in \overline{NE(X')}_{(K_{X'}+(r-1)L)<0}$.

By the above step Y and Y' are smooth; thus we can assume $Y' = \{x_n = 0\} \subset Y$.

By induction

$$f_*(\mathcal{O}_{X'}(-dbE) = (x_1^{s_1}\cdots x_{(n-1)}^{s_{(n-1)}} \mid s_1 + s_2 + \sum_{j=3}^{n-1} bs_j \geq db).$$

Proof

b) Consider the exact sequence on X

$$0 \to \mathcal{O}_X(-X' - dbE) \to \mathcal{O}_X(-dbE) \to \mathcal{O}_{X'}(-dbE) \to 0$$

Pushing it down via φ and using the Relative Kawamata-Viehweg Vanishing we have

$$0 \to f_*\mathcal{O}_X(-(d-1)bE) \xrightarrow{\cdot x_n} f_*\mathcal{O}_X(-dbE) \to f_*\mathcal{O}_{X'}(-dbE) \to 0.$$

The proposition follows by induction on n
$(f_*(\mathcal{O}_{X'}(-dbE) = (x_1^{s_1} \cdots x_{(n-1)}^{s_{(n-1)}} \mid s_1 + s_2 + \sum_{j=3}^{n-1} bs_j \geq db)))$
and on d
$(f_*(\mathcal{O}_X(-(d-1)bE) = (x_1^{s_1} \cdots x_n^{s_n} \mid s_1 + s_2 + \sum_{j=3}^{n} bs_j \geq (d-1)b))$

$r = (n-1)$

Consider a $K_X + \Delta^r$-MMP with $r = (n-1)$ (or $\geq (n-1)$) and let $R_i = \mathbb{R}^+[C_i]$ be a birational ray in the sequence.

Inductively construct a nef and big Cartier divisor L_i on X_i such that $rL_i \sim_{\mathbb{Q}} \Delta_i^{(n-1)}$:

1) $L_i C_i = 0$, otherwise, by the above Theorem, we have the contradiction $(n-1) \geq \dim F > r \geq (n-1)$.

2) Let $\varphi_i : X_i \to Y$ be the contraction associated with R_i. We have a Cartier divisor L'_{i+1} such that $\varphi^*(L'_{i+1}) = L_i$.

If φ is birational $(X_{i+1}, L_{i+1}) := (Y, L'_{i+1})$,

if φ is small $(X_{i+1}, L_{i+1}) := (X_i^+, \varphi^+(L'_{i+1}))$, where $\varphi^+ : X_i^+ \to Y$ is the flip.

the zero reduction

Proposition. Given a q.p. pair (X, L) it is possible to run a MMP which contracts all extremal rays on which L is zero and obtain a q.p. pair (X', L') which is birational equivalent to (X, L) and such that:

- either $K_{X'} + (n-1)L'$ is nef
- or (X', L') is a Mori space relative to $K_{X'} + (n-1)L'$ and L' is a (relatively) very ample Cartier divisor.

Definition. (X', L') is called a zero reduction of (X, L).

By very classical results in the second case the q.p. pair (X', L') is in a obvious finite list of examples: $(\mathbb{P}^n, \mathcal{O}(1))$, $(Q, \mathcal{O}(1))$, scrolls, del Pezzo.

Applications

Let (X, L) be a quasi-polarized variety and $g(X, L)$ be its sectional genus: $2g(X, L) - 2 = (K_X + (n-1)L) \cdot L \cdot \ldots \cdot L)$.

(if L is spanned it is the genus of a curve intersection of $n - 1$ general elements in $|L|$.

- Classification of pairs $g(X, L) \leq 0$ ($K_X + (n-1)L$ is not nef (therefore not pseudoeffective) and the zero reduction of (X, L) is among the above pairs

- Classification of pairs with $g(X, L) = 1$.

- Classification of pairs of minimal degree (i.e. $L^n = h^0(X, L) - n$).

First Reduction

Proposition-Part 1. Let (X, L) be a q.p. pair. There exists a q.p. pair (X'', L'') which is a $(K_X + \Delta^{n-2})$-MM and which can be obtained with the following procedure:

- Take a zero reduction (X', L').
- Contract, step by step, all Castelnuovo-Kawakita type extremal rays, such that $L'_{|E} = -bE_{|E}$; $\varphi' : X' \to X''$.
- Let $L'' := \varphi'_* L'$.

Definition The pair (X'', L'') is called a First Reduction of the pair (X, L).

First Reduction

Proposition-Part 2. Let (X, L) be a q.p. pair and let (X'', L'') be its First Reduction. Then

- either $K_{X''} + (n-2)L''$ is nef
- or $X'' \to Z$ is a Mori fiber space relatively to $K_{X''} + (n-2)L''$ and L'' is (relatively) very ample with one exception (del Pezzo manifold). In all cases there exists a divisor in $|L''|$ with good singularities.

Remark. The classification of the pairs in the second part, thank to the existence of a good section, is classical and reduces to the theory of algebraic surfaces. (Quadric fibration, del Pezzo manifolds, ...)

Applications

Theorem. Let $Y \subset \mathbb{P}^N$ be a non degenerate projective variety of dimension $n \geq 3$ of degree d and let $\tilde{L} := \mathcal{O}(1)_{|Y}$. Assume that $d < 2codim_{\mathbb{P}^N}(X) + 2$.

Then on a desingularization (X, L) the divisor $K_X + (n-2)L$ is not pseudoeffective.

Therefore $(Y, \mathcal{O}(1))$ is equivalent, via birational equivalence and first-reduction, to a q.p. pair (X'', L'') in the above Remark.

$(n-3) < \tau \leq (n-2)$

Let $\varphi : X \to Y$ be a birational contraction associated with an extremal ray $R = \mathbb{R}^+[C]$ on a q.p. pair, such that $L \cdot C > 0$ and $(n-3) < \tau \leq (n-2)$.
(These are the birational maps in a $K_X + \Delta^{n-3}$-MMP).

We have the following possibilities for φ.

1. φ contracts a divisor to a curve (it is a special case of the next theorem),
2. φ contracts a divisor to a point,
3. φ is a small contraction with exc. locus of dimension $(n-2)$.

In all cases we can apply the "base point free technique", find therefore a section $X' \in |L|$ and, by induction if possible, reduce to the case $n = 3$ where we have a complete classification.
Then we like to "lift" this classification and the examples to higher dimension.

Divisorial contractions with small fibers

Theorem. Let $f : X \to Z$ be a divisorial contraction associated to an extremal ray R in $\overline{NE(X)}_{(K_X+rL)<0}$, where $r \in \mathbb{N}^+$, such that $L \cdot C > 0$. Let E be the exceptional locus of f and set $C := f(E) \subset Z$.

Assume that all fibres have dimension less or equal to $r+1$.

1. Then there is a closed subset $S \subset Z$ of codimension al least 3 such that $Z' = Z \backslash S$ and $C' = C \backslash S$ are smooth, $\mathrm{codim}_{Z'} C' = r+2$ and $f' : X' = X \backslash f^{-1}(S) \to Z'$ is a weighted blow-up along C' with weight $\sigma = (1, 1, b \dots, b, 0, \dots, 0)$, where the number of b's is r.
2. Let $\mathcal{I}'$ be a σ-weighted ideal sheaf of degree b for $Z' \subset X'$ and let $i : Z' \to Z$ be the inclusion; let also $\mathcal{I} := i_*(\mathcal{I}')$ and $\mathcal{I}^{(m)}$ be the m-th symbolic power of $\mathcal{I}$. Then $X = \mathrm{Proj} \bigoplus_{m \geq 0} \mathcal{I}^{(m)}$.

Kawakita Contractions

The second case was treated for $n = 3$ in a series of paper by Kawakita.

We would like to extend Kawakita classification and show that φ is a weighted blow-up of a (possible singular) point.

We are trying to use Cox Rings of weighted blow-ups:
- they determine completely the blow-up,
- there are some new general results on the relation between the Cox Ring of a variety and of an ample section.

Small Contractions

The third case was treated for $n = 3$ by S. Mori and by S.Mori and J. Kollar.

At the moment we can extend to higher dimension the Francia's flip, i.e. the case where X has only points of index one and two.