Holomorphic Symplectic Manifolds

Marco Andreatta

joint project with J. Wiśniewski

Dipartimento di Matematica

Universitá di Trento

Definitions

An holomorphic symplectic manifold X
is a kähler manifold X with a holomorphic non degenerate closed form $\sigma \in H^0(X, \Omega^2_X)$
An irreducible holomorphic symplectic manifold X
is compact and $H^0(X, \Omega^*_X) = \mathbb{C}[\sigma]$
(eq. X is simply connected and $< \sigma >= H^0(X, \Omega^2_X)$).

Definitions

An holomorphic symplectic manifold X
is a kähler manifold X with a holomorphic non degenerate closed form $\sigma \in H^0(X, \Omega_X^2)$
An irreducible holomorphic symplectic manifold X
is compact and $H^0(X, \Omega_X^*) = \mathbb{C}[\sigma]$
(eq. X is simply connected and $< \sigma >= H^0(X, \Omega_X^2)$).

Calabi-Yau : projective mfds with $H^0(X, \Omega_X^*) = \mathbb{C} + \mathbb{C}\omega$, where ω is a generator for K_X.

Definitions

An holomorphic symplectic manifold X
is a kähler manifold X with a holomorphic non degenerate closed form $\sigma \in H^0(X, \Omega^2_X)$

An irreducible holomorphic symplectic manifold X
is compact and $H^0(X, \Omega^*_X) = \mathbb{C}[\sigma]$
(eq. X is simply connected and $< \sigma >= H^0(X, \Omega^2_X)$).

Calabi-Yau : projective mfds with $H^0(X, \Omega^*_X) = \mathbb{C} + \mathbb{C}\omega$, where ω is a generator for K_X.
Complex Tori : $\mathbb{C}^n/\Gamma$, $\Gamma < \mathbb{C}^n$ a discrete subgroup.

Definitions

An holomorphic symplectic manifold X
is a kähler manifold X with a holomorphic non degenerate closed form $\sigma \in H^0(X, \Omega^2_X)$
An irreducible holomorphic symplectic manifold X
is compact and $H^0(X, \Omega^*_X) = \mathbb{C}[\sigma]$
(eq. X is simply connected and $< \sigma >= H^0(X, \Omega^2_X)$).

Calabi-Yau : projective mfds with $H^0(X, \Omega^*_X) = \mathbb{C} + \mathbb{C}\omega$, where ω is a generator for K_X.
Complex Tori : $\mathbb{C}^n/\Gamma$, $\Gamma < \mathbb{C}^n$ a discrete subgroup.

Theorem (Bogomolov). Z kähler manifold with $c_1(Z) = 0$.
Up to an etale cover $Z' \simeq$ Tori x Calabi-Yau x Irr. Sympl.

Kummer construction

Following Kummer, Fujiki, Beauville, we take

Data:
*) A a complex torus of dimension d
*) $G < GL(r.\mathbb{Z})$ an irreducible representation of a finite subgroup; if d is odd we assume $G < SL(r.\mathbb{Z})$.

Kummer construction

Following Kummer, Fujiki, Beauville, we take

Data:

*) A a complex torus of dimension d

*) $G < GL(r.\mathbb{Z})$ an irreducible representation of a finite subgroup; if d is odd we assume $G < SL(r.\mathbb{Z})$.

Construction:

i) Consider the action of G on A^r and take the quotient variety $Y := A^r/G$.

ii) Take a crepant resolution $\pi : X \to Y$.

Kummer construction

Following Kummer, Fujiki, Beauville, we take

Data:

*) A a complex torus of dimension d

*) $G < GL(r.\mathbb{Z})$ an irreducible representation of a finite subgroup; if d is odd we assume $G < SL(r.\mathbb{Z})$.

Construction:

i) Consider the action of G on A^r and take the quotient variety $Y := A^r/G$.

ii) Take a crepant resolution $\pi : X \to Y$.

Remark: Crepant resolution means $K_X \simeq \pi^* K_Y$.

It is hard to find!!!

Kummer construction

Following Kummer, Fujiki, Beauville, we take

Data:
*) A a complex torus of dimension d
*) $G < GL(r.\mathbb{Z})$ an irreducible representation of a finite subgroup; if d is odd we assume $G < SL(r.\mathbb{Z})$.

Construction:
i) Consider the action of G on A^r and take the quotient variety $Y := A^r/G$.
ii) Take a crepant resolution $\pi : X \to Y$.

Remark: Crepant resolution means $K_X \simeq \pi^* K_Y$.
It is hard to find!!!

Final Output: A manifold with $K_X \simeq \mathcal{O}_X$ and $H^1(X, \mathbb{C}) = \mathbb{C}$, i.e. Calabi-Yau or Symplectic.

Examples

More specifically one can check

*) Finite subgroups of $SL(2,\mathbb{Z})$ acting on $A^2 = (\mathbb{C}/\Gamma)^2$. The quotient has rational double points, so there exist crepant resolution, we get Kummer surfaces (K3).

Examples

More specifically one can check

*) Finite subgroups of $SL(2,\mathbb{Z})$ acting on $A^2 = (\mathbb{C}/\Gamma)^2$. The quotient has rational double points, so there exist crepant resolution, we get Kummer surfaces (K3).

*) Finite subgroups of $SL(3,\mathbb{Z})$ acting on $A^3 = (\mathbb{C}/\Gamma)^3$. There exist crepant resolutions (Roan and others), we get Calabi-Yau 3-folds.

Examples

More specifically one can check

*) Finite subgroups of $SL(2,\mathbb{Z})$ acting on $A^2 = (\mathbb{C}/\Gamma)^2$.
The quotient has rational double points, so there exist crepant resolution, we get Kummer surfaces (K3).

*) Finite subgroups of $SL(3,\mathbb{Z})$ acting on $A^3 = (\mathbb{C}/\Gamma)^3$.
There exist crepant resolutions (Roan and others), we get Calabi-Yau 3-folds.

*) Finite subgroups of $Sp(2n,\mathbb{C})$ acting on $A^n = (\mathbb{C}^2/\Gamma)^n$.
due to Fujiki-Beauville.
Together with two sporadic examples of O' Grady they are the only examples of Irreducible Symplectic mfds.

Local symplectic contractions

Consider a local symplectic contractions $\pi : X \to Y$ where

- X is a symplectic manifold
- Y is an affine normal variety,
- π is a birational projective morphism with connected fibers.

Local symplectic contractions

Consider a local symplectic contractions $\pi : X \to Y$ where

- X is a symplectic manifold
- Y is an affine normal variety,
- π is a birational projective morphism with connected fibers.

In dimension 2 symplectic contractions are classical and they are minimal resolutions of Du Val singularities $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.
They are quotients of type $\mathbb{C}^2/H$ with $H < SL(2, \mathbb{C})$ a finite group.

Quotient symplectic singularities

For example take $G < Sp(2n, \mathbb{C})$, i.e. G preserves a symplectic form σ.
For any resolution $\pi : X \to \mathbb{C}^{2n}/G$ the form $\pi^*(\sigma)$ extends to a holomorphic two form on X (Beauville).
If it is non degenerate everywhere then π is a symplectic resolution (a symplectic contraction). This is equivalent to be crepant.

Quotient symplectic singularities

For example take $G < Sp(2n, \mathbb{C})$, i.e. G preserves a symplectic form σ.
For any resolution $\pi : X \to \mathbb{C}^{2n}/G$ the form $\pi^*(\sigma)$ extends to a holomorphic two form on X (Beauville).
If it is non degenerate everywhere then π is a symplectic resolution (a symplectic contraction). This is equivalent to be crepant.

Problem: describe $G < Sp(2n, \mathbb{C})$ which admit a symplectic resolution (even for $n = 2$).

Examples

Let S be a smooth surface. Then

$$Hilb^n(S) := S^{[n]}S \to (S)^n/\sigma_n := S^n(S)$$

is a crepant resolution; it is the blow-up of the diagonal

Examples

Take $H < SL(2, \mathbb{C})$ and let $S \to \mathbb{C}^2/H$ be the minimal desingularization (symplectic contraction).

Consider the composition

$$S^{[n]}S \to S^n(S) \to S^n(\mathbb{C}^2/H)$$

it is a crepant map.

It is the symplectic resolution of $S^n(\mathbb{C}^2/H) = \mathbb{C}^{2n}/G$ where $G = (H)^n \rtimes \sigma_n < Sp(2n)$.

This is the local $Hilb^n$ case of Beauville and Fujiki.

Examples

Consider the composition $S^{[n+1]}(\mathbb{C}^2) \to S^{n+1}(\mathbb{C}^2) \to \mathbb{C}^2$, where the last is $\tau : (a_1, ..., a_{n+1}), (b_1, ..., b_{n+1}) \to (\Sigma a_i, \Sigma b_i)$.

The restriction $X := \pi^{-1}(0,0) \to \tau^{-1}(0,0)$ is a crepant map.

It is the symplectic resolution of $\tau^{-1}(0,0) = \mathbb{C}^{2n}/G$ where $G = \sigma_{n+1} < Sp(2n)$.

This is the local Kum^n case of Beauville and Fujiki.

Symplectic resolution for $\mathbb{C}^n \oplus \mathbb{C}^{n*}$

Let $G < GL(n, \mathbb{C})$ a finite subgroup. G can be viewed as a subgroup $G < Sp(\mathbb{C}^n \oplus \mathbb{C}^{n*}) = Sp(\mathbb{C}^{2n})$ (the symplectic form preserved is the identity in $\mathbb{C}^n \oplus \mathbb{C}^{n*}$.

Symplectic resolution for $\mathbb{C}^n \oplus \mathbb{C}^{n*}$

Let $G < GL(n, \mathbb{C})$ a finite subgroup. G can be viewed as a subgroup $G < Sp(\mathbb{C}^n \oplus \mathbb{C}^{n*}) = Sp(\mathbb{C}^{2n})$ (the symplectic form preserved is the identity in $\mathbb{C}^n \oplus \mathbb{C}^{n*}$.

Theorem. (Bellamy) Let $G < GL(n, \mathbb{C})$; a symplectic resolution of $\mathbb{C}^{2n}/G$ exists iff G is one of the two groups above or G is the binary tetrahedral group, $BT = Q \rtimes \mathbb{Z}_3$, acting on $\mathbb{C}^4 = C^2 \oplus C^{2*}$.

Symplectic resolution for $\mathbb{C}^n \oplus \mathbb{C}^{n*}$

Let $G < GL(n, \mathbb{C})$ a finite subgroup. G can be viewed as a subgroup $G < Sp(\mathbb{C}^n \oplus \mathbb{C}^{n*}) = Sp(\mathbb{C}^{2n})$ (the symplectic form preserved is the identity in $\mathbb{C}^n \oplus \mathbb{C}^{n*}$.

Theorem. (Bellamy) Let $G < GL(n, \mathbb{C})$; a symplectic resolution of $\mathbb{C}^{2n}/G$ exists iff G is one of the two groups above or G is the binary tetrahedral group, $BT = Q \rtimes \mathbb{Z}_3$, acting on $\mathbb{C}^4 = C^2 \oplus C^{2*}$.

Remark. a)Lehn-Sorger described explicitly a local symplectic resolution for $\mathbb{C}^4/BT$.

b) Kummer construction applies for the first two but it does NOT apply for BT (i.e. there is no global symplectic resolution) (– ,Wisniewski).

Special properties

Remarks: Y has rational singularities, K_Y is trivial and π is crepant. All exceptional fibres are uniruled.

Special properties

Remarks: Y has rational singularities, K_Y is trivial and π is crepant. All exceptional fibres are uniruled.

Theorem (Wierzba - Namikawa) π is semismall, i.e. for every $Y \subset X$
$2codim(F) \geq codim(\pi(F))$.
If equality holds then F is called a maximal cycle.

Special properties

Remarks: Y has rational singularities, K_Y is trivial and π is crepant. All exceptional fibres are uniruled.

Theorem (Wierzba - Namikawa) π is semismall, i.e. for every $Y \subset X$
$2codim(F) \geq codim(\pi(F))$.
If equality holds then F is called a maximal cycle.

Theorem (Z. Ran- Wierzba) Let $f : \mathbb{P}^1 \to X$ be a non constant map whose image is a π exceptional curve. Then f deforms in a family (Hilb) of dimension at least $dim X + 1$.

McKay correspondence

Let $G < Sp(V)$ and for $g \in G : age(g) = 1/2codim(V)^g$.

Theorem (Batyrev-Kaledin- ...).
Assume there exists a symplectic resolution $\pi : X \to V/G$, then:
$dimH_{2i}(X, \mathbb{Q}) = \sharp\{$conj. classes of el. of G of age $= i\}$;
odd homology is zero.

McKay correspondence

Let $G < Sp(V)$ and for $g \in G : age(g) = 1/2 codim(V)^g$.

Theorem (Batyrev-Kaledin- ...).
Assume there exists a symplectic resolution $\pi : X \to V/G$, then:
$dim H_{2i}(X, \mathbb{Q}) = \sharp\{$conj. classes of el. of G of age $= i\}$;
odd homology is zero.

Moreover there exists a base of $H_{2i}(X, \mathbb{Q})$ given by maximal cycles which are counter-image of $(V)^g$ with $codim V^g = i$.

4-*dimensional case*

From now on we restrict to the case $dimX = 4$.

By semi-smallness $Exc(\pi)$ consists (possibly) of

- D_j exceptional divisors mapping to surfaces $\pi(D_j) := S_i \subset Y$
- T_k two dimensional special fibers $\pi(T_k) = pt := 0 \in Y$.

Remark. (Wierzba)
The general fiber F is a tree of rational curves.
Components of a general fiber are callled essential curves; if two stay in the same component D_j they are deforamtion equivalent.
The normalization of a two dimensional fiber is a rational surface.

Small contractions

Theorem
(Wierzba-Wisniewski, Cho-Myiaoka- Sheperd Barron).
If π is small (i.e. $D_j = \emptyset$) then the T_i are a finite numbers of disjoint $\mathbb{P}^2$ with normal bundle $T^*\mathbb{P}^2$.
(Mukai flop)

Chow Scheme

Let $C \subset \pi^{-1}(s), s \in S_i \setminus \{= 0\}$ be an essential curve and $\mathcal{V} \subset Chow(X/Y)$ an irreducible component, i.e. a Chow family of rational curve, containing it and such that the map $\mathcal{V} \to \tilde{S}_i \to S_i$ is dominant.

Consider the incidence diagram:

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "q"] \arrow[d] & & D_j \subset X \arrow[d] \\
\mathcal{V} \arrow[r] & \tilde{S}_i \arrow[r] & S_i
\end{tikzcd}$$

where $\tilde{S}_i$ is the normalization.

Chow Scheme

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "q"] \arrow[d] & & D_j \subset X \arrow[d] \\
\mathcal{V} \arrow[r] & \tilde{S}_i \arrow[r] & S_i
\end{tikzcd}$$

Proposition (Wierzba, Conde-Wisniewski, — - Wisniewski)
$\mathcal{V}$ is smooth and it has a holomorphic closed two form non degenerate (possibly) outside some (-1)-curves.
In particular $\tilde{S}_i$ has at most a double point singularity at 0 and $\mathcal{V}$ is a not necessarly minimal desingularization of $\tilde{S}_i$.
q is not of maximal rank on the locus over the (-1)-curve.

Mori Dream Spaces

Let $\pi : X \to Y$ be a projective morphism of normal varieties with connected fibers and $Y = Spec A$.
By $\mathcal{N}ef(X/Y) \subset N^1(X/Y)$ we understand the closure of the cone spanned by the classes of relatively-ample bundles
By $\mathcal{M}ov(X/Y) \subset N^1(X/Y)$ we understand the cone spanned by the classes of linear systems which have no fixed components.

Mori Dream Spaces

Assume that X is $\mathbb{Q}$-factorial and $Pic(X/Y)$ is a lattice (finitely generated abelian group with no torsion); let $N^1(X/Y) = Pic(X/Y) \otimes \mathbb{Q}$.
We say that X is a Mori Dream Space (MDS) over Y if:

1. $\mathcal{N}ef(X/Y)$ is the affine hull of finitely many semi-ample line bundles:
2. there is a finite collection of small $\mathbb{Q}$-factorial modifications (SQM) over Y, $f_i : X - \to X_i$ such that $X_i \longrightarrow Y$ satisfies the above assumptions and $\mathcal{M}ov(X/Y)$ is the union of the strict transforms $f_i^*(\mathcal{N}ef(X_i))$.
   X_i are called the SQM models.

More on MDS

Note that a version of the theorems of Hu-Keel works in the relative situation too.
In particular, the relative Cox ring, $Cox(X/Y)$, is a well defined, finitely generated, graded module $\bigoplus_{L \in Pic(X/Y)} \Gamma(X, L)$.
Moreover X is a GIT quotient of $Spec(\bigoplus_{L \in Pic(X/Y)} \Gamma(X, L))$ under the Picard torus $Pic(Y/X) \otimes \mathbb{C}^*$ action.

Symplectic contractions are MDS

Theorem. Let $\pi : X \to Y$ be a 4-dimensional local symplectic contraction.
Then X is a Mori Dream Space over Y.
Moreover any SQM model of X over Y is smooth and any two of them are connected by a finite sequence of Mukai flops.
In particular, there are only finitely many non isomorphic (local) symplectic resolution of Y.

Symplectic contractions are MDS

Theorem. Let $\pi : X \to Y$ be a 4-dimensional local symplectic contraction.
Then X is a Mori Dream Space over Y.
Moreover any SQM model of X over Y is smooth and any two of them are connected by a finite sequence of Mukai flops.
In particular, there are only finitely many non isomorphic (local) symplectic resolution of Y.

Proof

- Cone theorem holds (Mori, Kawamata)
- Base point free theorem (Kawamata, Shokurov)
- Existence of flops (Wierzba-Wisniewski)
- Termination (Matsuki).

Movable cone

$N_1(X/Y)$ denotes the vector space of 1-cycles proper over Y. We define $\mathcal{E}ss(X/Y)$ as the convex cone spanned by the classes of curves which are not contained in $\pi^{-1}(0)$

Theorem (— -Wisniewski, Altmann-Wisniewski)
$\mathcal{M}ov(X/Y) =< \mathcal{E}ss(X/Y) >^{\vee}$.
In particular the cone $\mathcal{M}ov(X/Y)$ is symplicial.

Movable cone

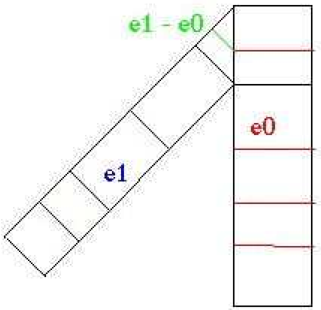
$N_1(X/Y)$ denotes the vector space of 1-cycles proper over Y. We define $\mathcal{E}ss(X/Y)$ as the convex cone spanned by the classes of curves which are not contained in $\pi^{-1}(0)$

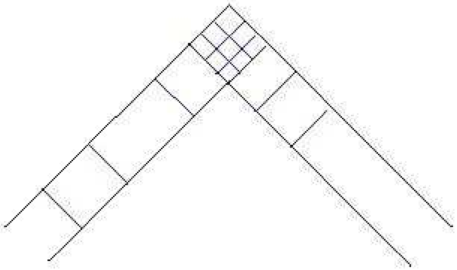
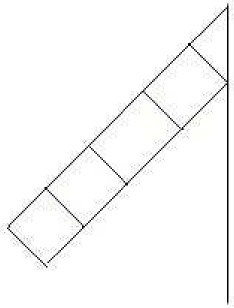
Theorem (— -Wisniewski, Altmann-Wisniewski)
$\mathcal{M}ov(X/Y) =< \mathcal{E}ss(X/Y) >^{\vee}$.
In particular the cone $\mathcal{M}ov(X/Y)$ is symplicial.

Theorem The subdivision of $\mathcal{M}ov(X/Y)$ into the subcones $\mathcal{N}ef(X_i/Y)$ is done by hyperplanes, corresponding to small contractions.
(that is the internal walls are of the type $C \cap \mathcal{M}ov(X/Y)$ where C is an hyperplane in $N^1(X/Y)$).

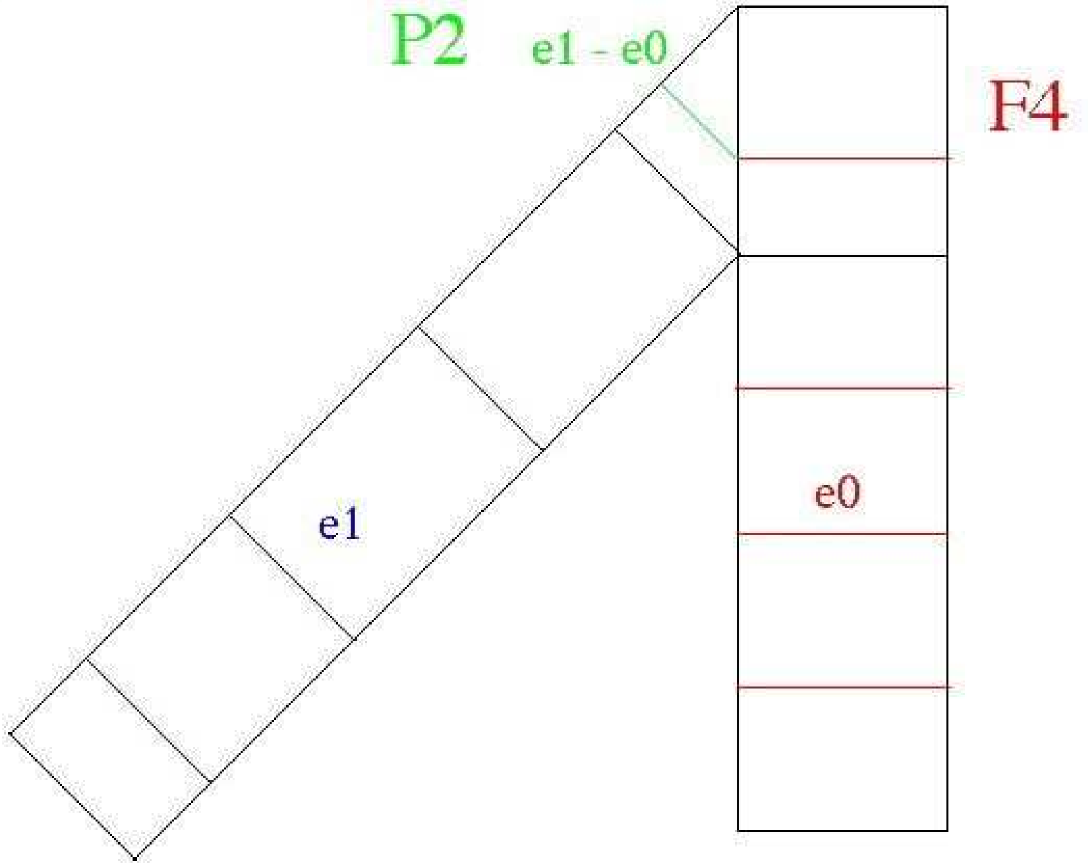
Semi direct product

$X = S^{[2]}$



S	$S \times S$	$(S \times S)/Z = S^{(2)}$
		
C^2/Z_2	$C^2/Z_2 \times C^2/Z_2$	$(C^2/Z_2 \times C^2/Z_2)/Z_2$

Semi direct product



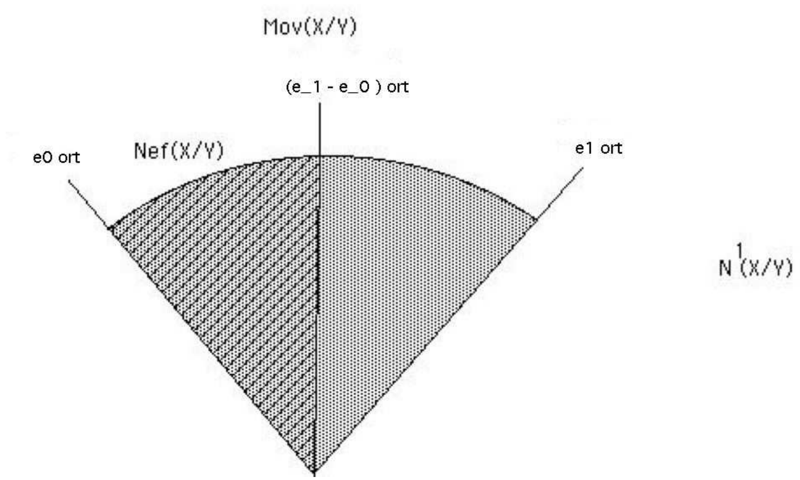
The resolution $X \to \mathbb{C}^4/(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2$ contains:
a reducible divisor $D = D_0 \cup D_1$,
the fiber over 0 which is $F_4 \cup \mathbb{P}^2$.

Semi direct product: Chow

The surface S has a singularity of type A_1
$\mathcal{V}_0$ is the minimal resolution, the curve in $\mathcal{U}_0$ over the -2 correspond to curves in F_4.
$\mathcal{V}_1$ is also the minimal resolution and the curve in $\mathcal{U}_1$ over the -2 correspond to the splitting curves in $F_4 \cup \mathbb{P}^2$.

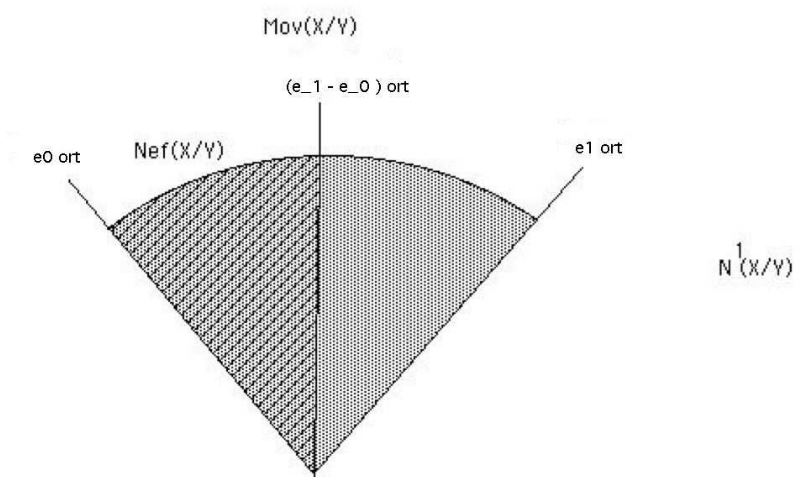
Semi direct product: MDS

$Pic(X/Y) = \mathbb{Z}^2 =$ and $NE(X/Y) =< e_0, e_1 - e_0 >$, where $e_1 - e_0$ is a line in $\mathbb{P}^2$.



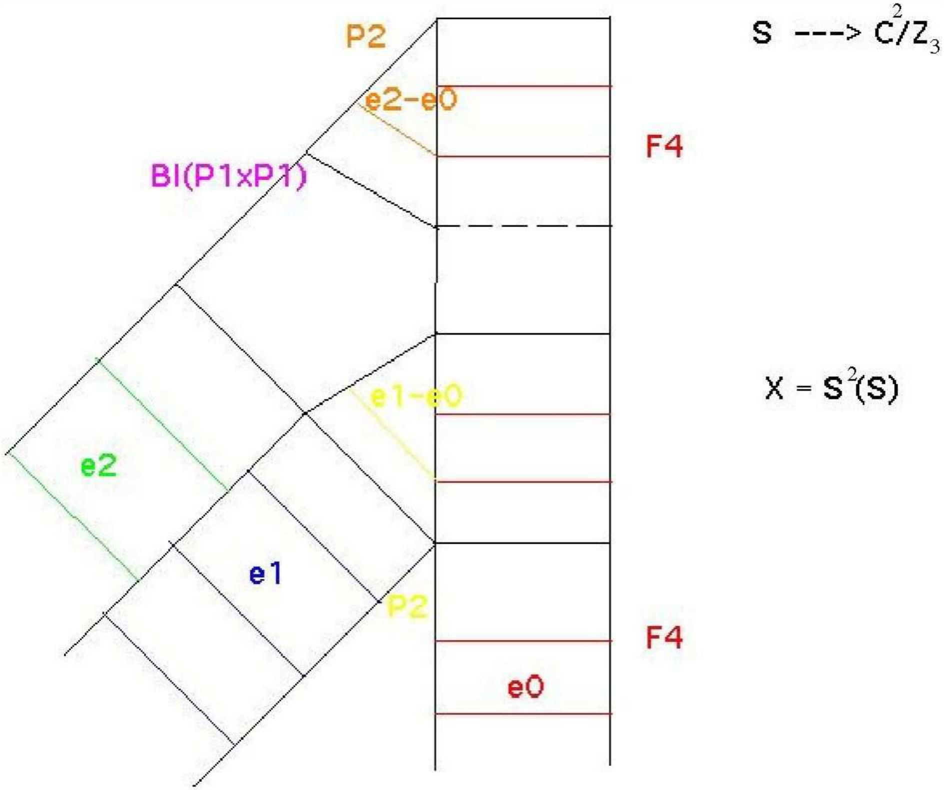
Semi direct product: MDS

$Pic(X/Y) = \mathbb{Z}^2 =$ and $NE(X/Y) =< e_0, e_1 - e_0 >$, where $e_1 - e_0$ is a line in $\mathbb{P}^2$.



There are two symplectic resolution of $\mathbb{C}^4/(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2$. They are symmetric and the flop passes from one to the other.

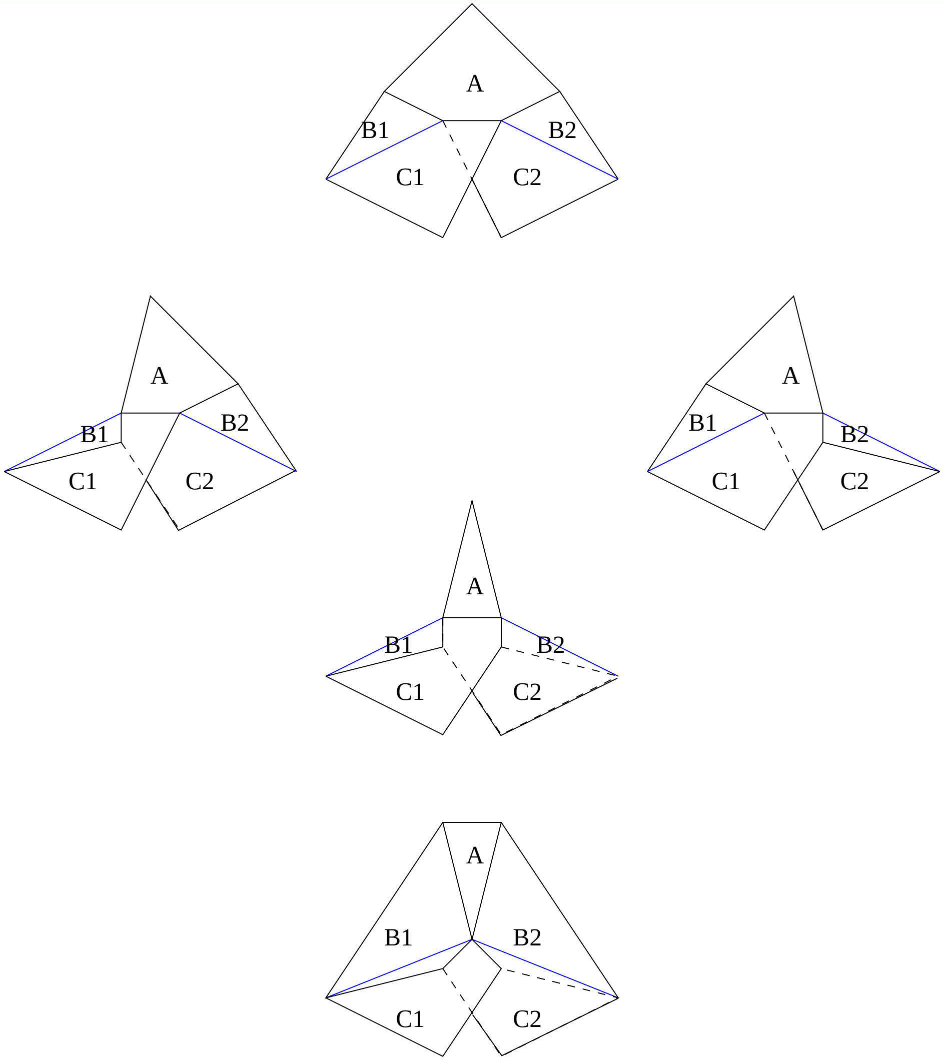
Semi direct product: resolution



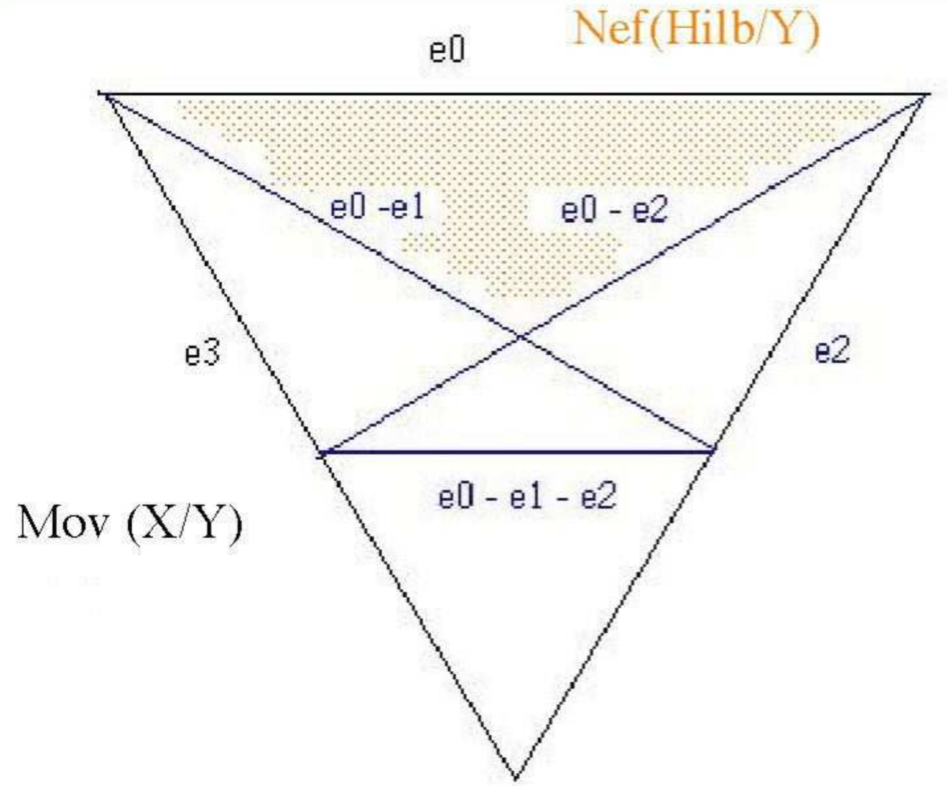
The resolution $X \to \mathbb{C}^4/BT$ contains:
a reducible divisor $D = D_1 \cup D_2$,

the fiber over 0 which is $F_4 \cup F_4 \cup Bl_p(\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{P}^2 \cup \mathbb{P}^2$.

Semi direct product: resolution



Semi direct product: MDS



There are five different symplectic resolutions of $\mathbb{C}^4/(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$.

Movable cone for $(\mathbb{Z}_n)^2 \rtimes \mathbb{Z}_2$

Theorem Let $X \to \mathbb{C}^4/(\mathbb{Z}_n)^2 \rtimes \mathbb{Z}_2$ a symplectic resolution. The exceptional locus consists of n divisor, $D_0, D_1, ..., D_{n-1}$, and $(n+2)(n-1)/2$ two dimensional fibers. Let e_i be an essential curve in D_i.
Then $Mov(X/Y) =< e_0, e_1, ..., e_{n-1} >^{\vee}$.
The division of $Mov(X/Y)$ into Mori chambers is defined by hyperplanes $\lambda_{ij}^{\perp}$ for $1 \leq i \leq j \leq n$ where $\lambda_{ij} = e_0 - (e_i + ... + e_j)$.

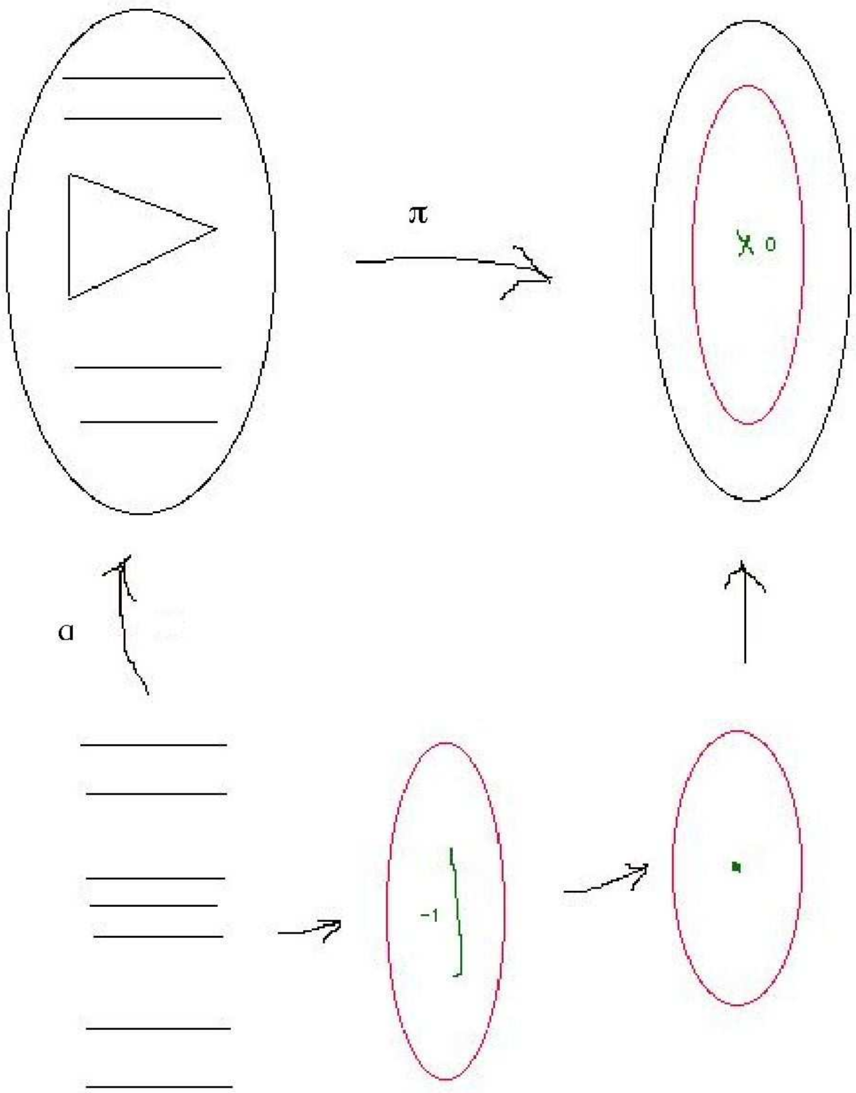
To each chamber it corresponds a symplectic resolution; they give all the symplectic resolutions.

σ_3 *example*

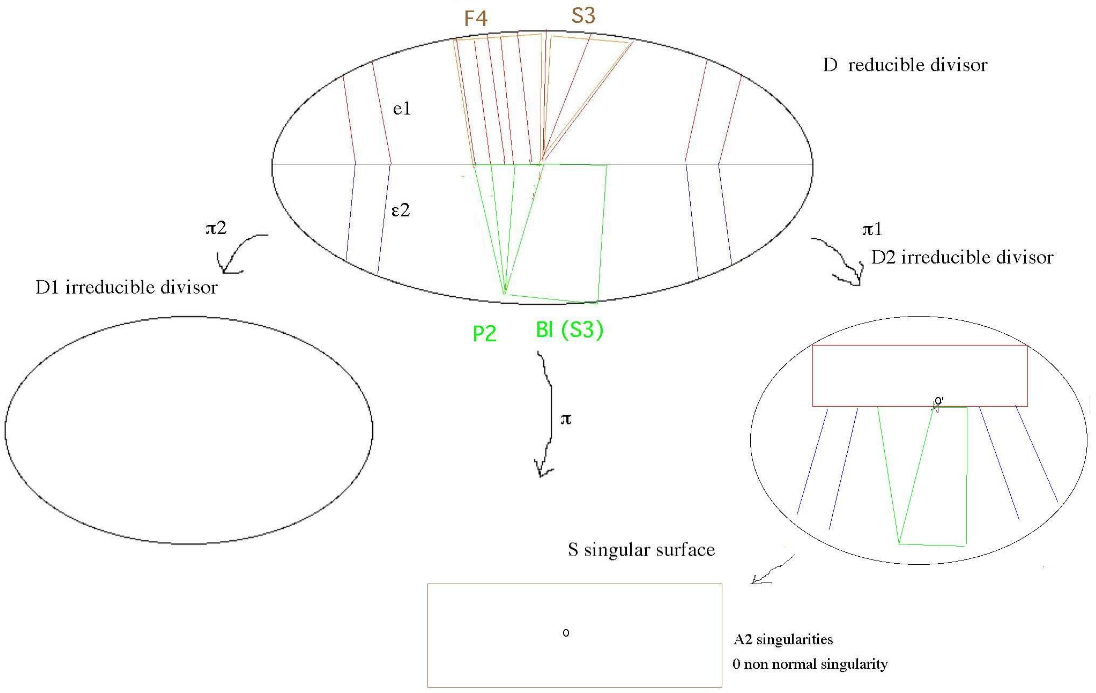
The resolution $X \to \mathbb{C}^4/\sigma_3$
contains an irr. divisor D
a fiber over 0 equal to S_3.

The surface $\tilde{S}$ is smooth,
$\mathcal{V}$ is the blow-up of a point,
the curves in $\mathcal{U}$ over the -1
correspond to lines in S_3.

$Pic(X/Y) = \mathbb{Z}$
and this is the unique
symplectic resolution
of $\mathbb{C}^4/\sigma_3$.



Binary tetrahedral



The resolution $X \to \mathbb{C}^4/BT$ contains:
a reducible divisor $D = D_1 \cup D_2$,
the fiber over 0 which is $S_3 \cup F_4 \cup Bl_p(S_3) \cup \mathbb{P}^2$.
π_1 contracts D_1 as above, π_2 is the small contraction of $\mathbb{P}^2$.

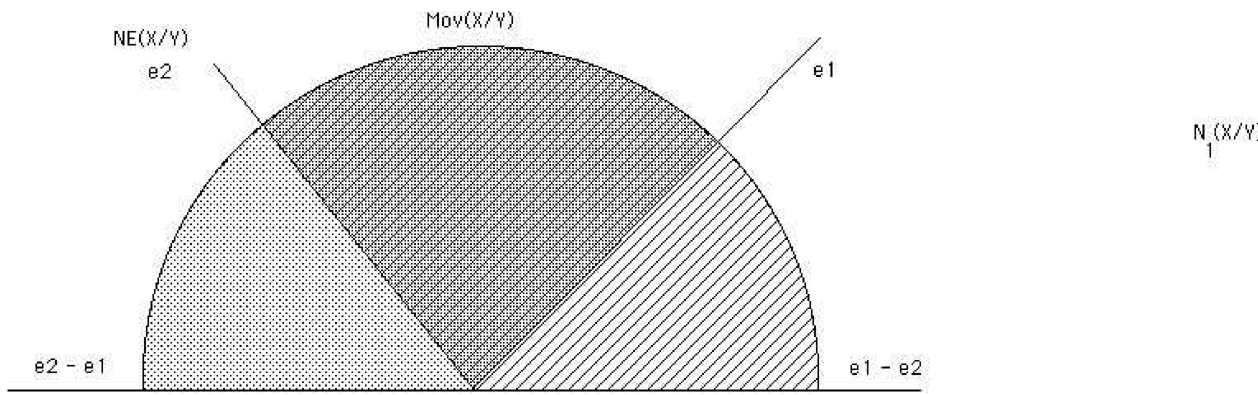
Binary tetrahedral: Chow

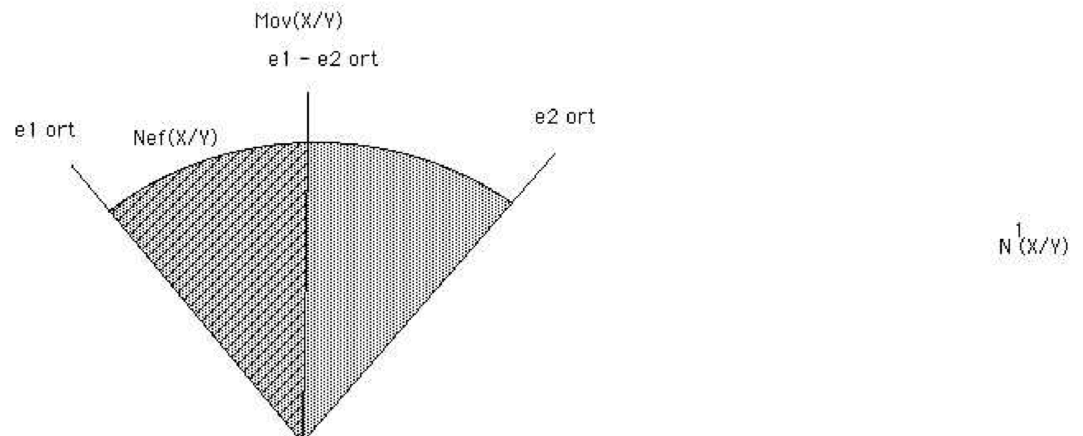
The surface $\tilde{S}$ has a $\mathbb{A}_1$ singularity.

$\mathcal{V}_1$ and $\mathcal{V}_2$ are both non minimal resolution of this singularities. In each case the fiber contains two rational curves meeting transversally: a (-1) curve, corresponding to curves in S_3 respectively $Bl_p(S_3)$, and a (-3)-curve.

Binary tetrahedral: MDS

$Pic(X/Y) = \mathbb{Z}^2 =$ and $NE(X/Y) =< e_1, e_2 - e_1 >$, where $e_2 - e_1$ is a line in $\mathbb{P}^2$.





Binary tetrahedral: resolutions

There are two symplectic resolution of $\mathbb{C}^4/BT$. They are symmetric and the flop passes from one to the other.

A conjecture

Conjecture Let $\pi : X \to Y$ be a local symplectic contraction on a smooth projective manifold of dimension 4. Assume that it is elementary, i.e. that $Pic(X/Y) \simeq \mathbb{Z}$. Then it is one of the following:

- the small contraction.
- the symplectic resolution of $\mathbb{C}^4/\sigma_3$
- the symplectic resolution of $\mathbb{C}^4/\mathbb{Z}_2$.