Note for a conference

On manifolds whose tangent bundle contains an ample subbundle

by Marco Andreatta
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Let X be a complex projective manifold of dimension n and let E be a vector bundle of rank r which is a subsheaf of the tangent sheaf TX, $E \hookrightarrow TX$.

Theorem. *If E is ample then $X \cong \mathbf{P}^n$ and $E \cong \mathcal{O}(1)^{\oplus r}$ or $E \cong T\mathbf{P}^n$.*

History.

R.Hartshorne conjectured that $\mathbf{P}^n$ is the only manifold whose tangent bundle is ample.

S. Mori proved the conjecture (i.e. the theorem for $r = n$) in a celebrated paper, which contained an amazing proof of the existence of rational curves on Fano manifolds.
Projective manifolds with ample tangent bundles, Ann. Math. **110** *(1979), 593-606*

J. Wahl Building up on Mori's work proved the theorem for $r = 1$
A cohomological characterization of P^n, Inv. Math. **72** *(1983), 315-322*

F. Campana and T. Peternell proved the theorem for $r = n-1,\ \ n-2$ and conjecture the general theorem.
Rational curves and ampleness properties of tangent bundle of algebraic varieties, Manuscr. Math., **97** *(1998), 59-74*

There are many further generalization to the above theorem, a nice one whould be the following conjecture, due again to F. Campana and T. Peternell.

Conjecture. *A Fano manifold with nef tamgent bundle is a rational homogeneous space.*

In the conference I will sketch the proof of the theorem. It applies many fundamental results and technique of the classification of higher dimensional projective varieties, based on rational curves on X, and developed in the last 30 years after Mori's paper.

A basic reference is the book of J. Kollár, which we will denote by [Ko]:
Rational Curves on Algebraic Varieties, Ergebnisse der Math. **32** *(1995), Springer Verlag.*

Prologue.

We start with a very powerfull criterium for uniruledness, which fits with our assumptions.
Theorem of Y. Miyaoka (improved by **Shepherd-Barron**).
If TX is not generically seminegative (i.e. there exists a subsheaf $E \hookrightarrow TX$ such that $c_1(E).C > 0$ for a curve C) then X is uniruled.

So we can choose a closed irreducible component $V \subset \mathrm{Hom}(\mathbf{P}^1, X)$ which covers X (that is $\mathrm{Locus}(V) = X$).
Moreover the family can be taken generically unsplit and such that the general $f \in V$ is a **minimal free morphism**, i.e. $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus d} \oplus \mathcal{O}^{\oplus(n-d-1)}$ where $d = \deg(f^*(-K_X)) - 2$
(see [Ko], IV. 2.4, 2.8, 2.9 and 2.10).
Let $V_x := V \cap \mathrm{Hom}(\mathbf{P}^1, X; 0 \mapsto x)$ and $(f^*TX)^+_p = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus d}$.

For a general, and thus since E is ample for any, $f \in V$ the pull-back f^*E is isomorphic either to $\mathcal{O}(1)^{\oplus r}$ or to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$.
In particular the family of curves parametrized by V is unsplit.

The family V defines a relation of **rational connectedness with respect to** V, rc$V$ relation for short, in the following way:
$x_1,\ x_2 \in X$ are in the rc$V$ relation if there exists a chain of rational curves parametrized by morphisms from V which joins x_1 and x_2.
The rc$V$ relation is an equivalence relation and its equivalence classes can be parametrized generically by an algebraic set. More precisely, we have the following result due to Campana and, independently, to Kollár-Miyaoka-Mori.

Theorem. *There exist an open subset $X^0 \subset X$ and a proper surjective morphism with connected fibers $\varphi^0 : X^0 \to Z^0$ onto a normal variety, such that the fibers of φ^0 are equivalence classes of the rc$V$ relation.*

We call the morphism φ^0 an rc$V$ fibration.
For our pourpose we have the following key observation.

Lemma. *Let X, E and V be as above and $\varphi^0 : X^0 \to Z^0$ is an rc$V$ fibration. Then E is tangent to a general fiber of φ^0. i.e.the injection $E_{|X_g} \to TX_{|X_g}$ factors via $E_{|X_g} \hookrightarrow TX_g$, where X_g is a general fiber.*

Proof. Since $\mathrm{Locus}(V) \subset X_g$ then $f^*(E) \subset (f^*TX)^+_p \subset (f^*TX_g)_p$ for every $p \in \mathbf{P}^1 \setminus \{0\}$ (see [Ko] II.3.4).

The proof it is now divided in two parts.

Part A Z_0 is just a point, i.e. X **is rationally connected (with the respect to** V**).**

Proposition. *The Picard number $\rho(X)$ is 1 (in particular X is Fano and $PicX = \mathbf{Z}$).*

Proof. This is a variation of Mori bend and break (see [Ko] IV.3.13.3).

First we have the case $f^*E = \mathcal{O}(1)^{\oplus r}$; this is reduced to (a special case of) Wahl theorem by the following proposition.

Proposition. *Suppose moreover that there exists an integer a such that for any $f \in V$ we have $f^*E = \mathcal{O}(a)^{\oplus r}$. Then there exists a (uniquely defined) line bundle L over X such that $\deg f^*L = a$ and $E \cong L^{\oplus r}$.*

Proof. Consider the projectivisation $p : \mathbf{P}(E) \to X$ and let $\mathcal{O}(1) = \mathcal{L}$.
For any $f \in V$ and $y \in p^{-1}(f(0))$ we have a unique lift-up $\hat{f} : \mathbf{P}^1 \to \mathbf{P}(E)$ such that $p \circ \hat{f} = f$ and $\deg(\hat{f}^*(\mathcal{L})) = a$, and $\hat{f}(0) = y$.
Let $\hat{V} \subset \mathrm{Hom}(\mathbf{P}^1, \mathbf{P}(E))$ an irreducible component which parameterizes these lift-ups and that dominates V, via the natural morphism $p_* : \mathrm{Hom}(\mathbf{P}^1, \mathbf{P}(E)) \to \mathrm{Hom}(\mathbf{P}^1, X)$ defined by $p_*(\hat{f}) = p \circ \hat{f}$.

Claim. *The morphism $p_* : \hat{V} \to V$ is proper and thus surjective, moreover $\hat{V}$ is an unsplit family.*

Let then consider the rc$\hat{V}$ fibration of $\mathbf{P}(E)$ and let $Y \subset \mathbf{P}(E)$ be a general fiber of this fibration.
The restriction map $p_Y : Y \to X$ is surjective and, since $\rho(Y) = 1$, it has no positive dimensional fiber, so it is a finite morphism.
We then prove that Y is actually a section of p and the proposition can be proved by induction.

If $f^*E = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$ we will reduce the theorem to (a special case of) Mori's theorem.

Fix a general $x \in X$ and consider the *derivative map*

$$\Phi_x : V_x \to P(T_xX) = P((f^*TX)_0)$$

which is defined at $[f] \in V_x$, if f is an immersion at 0, by $\Phi_x([f]) = [(Tf)_0(\partial/\partial t)]$.
(In the formula $Tf : T\mathbf{P}^1 \to f^*TX$ is the tangent map and T_xX is identified naturally, via f^*, with $(f^*TX)_0$ and P we denote the "natural projectivisation" (that is vector spaces modulo homotheties).)
This was first defined in the Mori's paper, later studied by Hwang-Mok and Kebekus.
We define $S_x \subset P(T_xX)$ as the closure of the image of the map Φ_x and we call it **tangent cone of curves from V at the point x.**
(J.-M.Hwang and N. Mok call this variety of minimal rational tangents, the name follows from the fact that S_x is (at least around $[f]$) the tangent cone to Locus(V_x)).

Lemma. *The projectivised tangent space of the tangent cone S_x at $\Phi_x([f])$ is equal to $P((f^*TX)_0^+) \subset P((f^*TX)_0) = P(T_xX)$.*

Proof. By [Ko] II.3.4.

Lemma. *If for a general $[f] \in V$ the tangent map $Tf : T\mathbf{P}^1 \to f^*TX$ factors to an injection $T\mathbf{P}^1 \hookrightarrow f^*\mathcal{E}$ (if $f^*E = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$), then $(f^*TX)^+ \hookrightarrow f^*\mathcal{E}$.*

Proof. We choose a general f which is an immersion at $0 \to x$. Then $\Phi_x([f]) \in P(\mathcal{E}_x) = P((f^*\mathcal{E})_0) \subset P(T_xX) = P((f^*TX)_0)$ and the same holds for morphisms in a neighborhood of $[f]$ in V_x. Thus around $\Phi_x([f])$ the tangent cone S_x is contained in $P(\mathcal{E}_x) = P((f^*\mathcal{E})_0)$, so is its tangent space $P((f^*TX)_0^+)$.

To conclude the, by the lemma we have the inclusion $(f^*TX)^+ \subset f^*E$ and hence that $f^*E = (f^*TX)^+$
Therefore $\deg(f^*E) = \deg(f^*(-K_X))$ and since $\rho(X) = 1$ it follows that $\det(E) = -K_X$.
The embedding $E \hookrightarrow TX$ gives rise to a non-trivial morphism $\det(E) \to \Lambda^r TX$ and thus to a non-zero section of $\Lambda^r TX \otimes K_X$. We use dualities to have the equalities:

$$h^0(X, \Lambda^r TX \otimes K_X) = h^n(X, \Omega_X^r) = h^r(X, \Omega_X^n) = h^r(X, K_X) = h^{n-r}(X, \mathcal{O}_X)$$

and, since X is Fano, the latter number is non-zero only if $r = n$. Thus $\Lambda^r TX \otimes (\det E)^{-1} \cong \mathcal{O}_X$ so $E \hookrightarrow TX$ is nowhere degenerate, hence an isomorphism, i.e $E = TX$.

Part B: $dim Z_0 \geq 1$.

Let $\varphi^0 : X^0 \to Z^0$ the rcV fibration and let X_g a general fiber: by the previous result X_g is isomorphic to $\mathbf{P}^k$ and E_{X_g} is either $TX_g = T\mathbf{P}^k$ or $\mathcal{O}(1)^{\oplus r}$.
We can shrink Z^0 and X^0 so that $\varphi^0 : X^0 \to Z^0$ is a projective space bundle in étale or analytic topology.

Claim We can extend φ^0 to a projective bundle space $\varphi' : X' \to Z'$ such that $codim(X \setminus X') \geq 2$.

If so we then take a general smooth projective curve B' in X which is contained in X' and is not contained in fibers of φ' and we consider the $\mathbf{P}^k$ bundle

$$f^* : X_B = (\varphi')^{-1}(B) \to B = \varphi'(B')$$

and the ample vector bundle $E_{|X_B}$. Since $E_{|X_g} \hookrightarrow TX_g$, one can actually see that $E_{|X_B}$ is a locally free subsheaf of $T_{X_B/B}$.
We arrive to a contradiction by applying the following result, which is due to Campana and Peternell.

Lemma. *Let X be a n-dimensional projective manifold, $\varphi : X \to Y$ a $\mathbf{P}^k$ bundle ($k < n$) of the form $X = \mathbf{P}(V)$ with a vector bundle V on Y. Then the relative tangent sheaf $T_{X/Y}$ does not contain an ample locally free subsheaf*

To prove the claim we do the following:

1) Take $\tilde{Z}$ an irreducible component of $\mathrm{Hilb}(X)$ such that $X_g \in \tilde{Z}$. Over $\tilde{Z}$ there exists a universal flat family $\tilde{X} \subset \tilde{Z} \times X$ with projections $\tilde{\varphi} : \tilde{X} \to \tilde{Z}$ and $\tilde{\beta} : \tilde{X} \to X$.
The family $\tilde{\varphi} : \tilde{X} \to \tilde{Z}$ extends $\varphi^0 : X^0 \to Z^0$.

2) We set $X^* = \tilde{X} \setminus E(\tilde{\beta})$ then, via $\tilde{\beta}$, we have inclusion $X^* \subset X$ and $codim(X \setminus X^*) \geq 2$.
Let φ^* be the restriction of $\tilde{\varphi}$ to X^* and let $Z^* = \varphi^*(X^*)$.

Lemma. *The morphism $\varphi^* : X^* \to Z^*$ is proper and a $\mathbf{P}^k$-bundle (in the analytic topology) outside a subset of codimension ≥ 2.*

Proof. The last part follows from a result of Fujita.