Special rays in the Mori cone of a projective variety

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1 Introduction

Let X be a smooth n-dimensional projective variety over an algebraically closed field k of arbitrary characteristic.
In 1978 Mori proved the so called Frenkel-Hartshorne conjecture who states the following:

Theorem 1.1 *T_X is ample if and only if $X = \mathbb{P}^n$.*

in the case of characteristic zero the Frenkel formulation is the following:

Theorem 1.2 *A compact complex manifold X admits a kähler metric whose holomorphic bisectional curvature is positive iff $X = \mathbb{P}^n$.*

Recently, together with J. A. Wiśniwski, we proved the following generalization of the Mori's Theorem.

Theorem 1.3 *Let E be any rank r vector bundle which is a subsheaf of the tangent sheaf T_X. E is ample if and only if $X = \mathbb{P}^n$.*

If $r = 1$ the Theorem was proved by J. Wahl, wether in the cases $r = n - 1$ and $n - 2$ by F. Campana and T. Peternell. This will not however be the topic of this conference.

In the case of curves, i.e. $n = 1$, the situation is optimal, via the Riemann uniformization theorem, and one can always find a metric on T_X which has constant curvature. In particular if it is negative then $X = \mathbb{P}^1$.
In higher dimension the situation is more intricated; for instance there are trivial examples of manifolds with curvature which can not be of definite sign.
Namely, given any manifold Y we can take the *blow-up* of it along a smooth subvariety $Z \subset Y$, $f : X = Bl_Z(Y) \longrightarrow Y$, or the *projectivization of the total space of a vector bundle* $\mathcal{E}$ on Y, $f : X = \mathbb{P}(\mathcal{E}) \longrightarrow Y$; in both cases we have that X contains some projective spaces and $-K_X$ is positive on the line of this spaces.

A key fact in the proof of Mori was the following: the assumption implies the existence of a curve $C \subset X$ with $(-K_X) \cdot C > 0$ (actually all curves satisfy this property since $-K_X := detT_X$ is ample). Then, using the formula

$$dimHom_{[f]}(C, X) \geq -K_X.C + n\chi(\mathcal{O}(C))$$

(which comes from the theory of deformation and Riemann-Roch theorem for the curve C) he proved the existence of rational curves through any point of X with negative intersection with K_X (a clever argument in positive characteristic is needed).

After this, the idea of Mori and others, in order to generalize the uniformization theorem in higher dimension, was to replace the hypothesis on positivity of the bundle T_X with the non nefness of K_X.

We recall that a line bundle H on X is said to be *numerically eventually free* or *nef* if $HC \geq 0$ for every curve $C \subset X$ (in other words if $H \geq 0$ on $\overline{NE(X)}$). The non nefness of K_X means therefore the fact that there are curves on X on which $detT_X$ is strongly positive (unfortunately there is a change of sign since $K_X = -detT_X$); this is a more natural request in higher dimension, as the previous examples told us.
The *Minimal Model Program* or the *Mori theory* consider the problem of finding out those negative curve with respect to K_X, describe them in some sense and eventually get rid of them, via birational trasformations, in order to get a so called *Minimal Model* i.e. a variety on which K_X is nef (therefore a variety which is non positively curved anywhere).
For this the first main theorem of Mori is the so called *cone theorem*. To introduce it we need to define $N_1(X)$ to be the vector space of 1-cycles on X modulo numerical equivalence and $\overline{NE(X)}$ to be the closure of the cone of effective 1-cycles (this is usually called the Kleiman-Mori cone). By definition, a *rational curve* is an irreducible, reduced curve whose normalization is $\mathbb{P}^1$.

Theorem 1.4 *(S. Mori) The part of the Mori-Kleiman cone $\overline{NE(X)}$ which stays in the negative semi-space defined by the linear form K_X is polyhedral and it is spanned by a countable number of extremal rays. Each extremal ray is spanned in $N_1(X)$ by a rational curve C_i with minimal degree with respect to K_X and such that $0 < -C_iK_X \leq dimX + 1$. (Moreover, moving an ϵ away from the hyperplane (in the negative direction) the number of extremal rays become finite.)*

May be the best way to illustrate the theorem is by a picture: (draw it!)

2 First Part; special rays

So the task at this point is to describe the rays, R_i, their locus, i.e. $Locus(R) =$ the set of points which are on a curve in R, and eventually find a map $f : X \longrightarrow Z$ which contracts all points in $Locus(R)$.
First of all we observe that we have two possibilities, namely $Locus(R) = X$ or $Locus(R) \subset X$. It is possible to prove that the last case happens if and only if there is an irreducible divisor $E \subset X$ such that $R.E < 0$. In this case we therefore say that the ray R is not nef; in particular Locus$(R) \subset E$.
If R is nef then an easy consequence of the Mori theory is that the variety X is uniruled, that is it is not only covered by curves in R but also by rational curves in R.
We will call the rational curve C generating a ray R minimal with respect to the intersection with $-K_X$ a *minimal extremal rational curve*. For such a curve C the integer $l(R) = -K_X.C$ is called the *length of the ray R* and the inequality $l(R) \leq n+1$ in the theorem can be made slightly more general in the following sense.
We will denote by $V \subset \mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ a closed irreducible subvariety, which is closed under $\mathrm{Aut}(\mathbb{P}^1)$ and which contains C (by the minimality of C we have

that V is an *unsplit family of rational curves* (see [Kol96, IV.2.]).)
We can define Locus(V) to be the image in X of the natural morphism associated to the family of rational curves V and Locus($V, 0 \to x$) to be the image of $V_x := \{f \in V : f(0) = x\}$ (since V is unsplit Locus(V) is a proper closed subscheme). Note that Locus(V) $\subset Locus(R)$.
Actually one can always find a *generically unsplit family* V such that $loc(V) = Locus(R)$ but we will not need it here. (By an easy argument, based on the existence of a finite number of family in the hilbert scheme, one can actually find a family which covers $Locu(R)$ and then one uses [Kol96, IV.2.4].)
We have the following fundamental inequality, due to P. Ionescu and J. Wiśniewski, coming out from the fundamental construction of S. Mori.

Proposition 2.1 *[Kol96, IV.2.5 and 2.6.] In the above notation and for any $x \in Locus(V)$ we have*

$$\dim Locus(V) + \dim Locus(V, 0 \to x) + 1 = \dim V \geq \dim X + l(R).$$

We will say that a ray R is **special** if its length is very large. In particular if it is maximal, namely $l(R) = n + 1$, we see from 2.1 that R has to be a nef ray and thus X is uniruled.
We have actually the following conjecture

Conjecture 2.2 X has a ray of length $n+1$ if and only if $X = \mathbb{P}^n$

This is a natural extension of the F-H conjecture in the frame of the MMP. Of course it implies the F-H conjecture and it can also be formulated with an apparently stronger hypothesis:

Conjecture 2.3 For every curve $C \subset X$, $-K_X{\cdot}C \geq n+1$ if and only if $X = \mathbb{P}^n$

(note that if X has a ray of length $n+1$ it is straightforward to prove that $Pic(X) = \mathbb{Z}$ and thus that $-K_X{\cdot}C \geq n+1$ for every curve $C \subset X$, see [Kol96, IV.3.13.3]).

The best result towards this conjecture is at the moment the following by Kachi and Kollár [KK00] :

Theorem 2.4 *$X = \mathbb{P}^n$ if and only if X has a ray of length $n+1$ and $(K_X)^n \geq (n+1)^n$*

This is proved with the next technical

Lemma 2.5 *[ABW93, Lemma1] if $char(k) = 0$ and [KK00, Lemma 8] if $char(k) > 0$. Let X be a normal, projective variety and $x \in X$ a smooth point. Let H be an ample Cartier divisor on X and let $\{C_t : t \in T\}$ be an $(n-1)$-dimensional family of curves through x. Assume that $(C_t.H) = 1$ and, if $char(k) > 0$ that $(-K_X.H^{n-1}) > (n-1)(H^n)$. Then $X \cong \mathbb{P}^n$, $H = \mathcal{O}_{\mathbb{P}^n}(1)$ and the C_t are lines through x.*

Scketch of the proof if $char(k) = 0$.....

Remark 2.6 If $char(k) = 0$ one can assume $(C_t.H) = 2$ instead of $(C_t.H) = 1$ and the result is the same (this was proved by Kebekus)

Proof. The Theorem follows from the lemma since by assumption $Pic(X) = \mathbb{Z}$, as noticed above, and thus $H := 1/(n+1)(-K_X)$ satisfies the assumption of the lemma; one has to note that the lemma remains true if H is just a $\mathbb{Q}$-Cartier divisor such that $H^n \geq 1$ and $H \cdot C \geq 1$)

On the other hand if the ray R is a non nef ray the inequality in 2.1 says that $l(R) \leq n-1$. In this case we have an optimal description of the ray.

Theorem 2.7 *Let X be a smooth n-dimensional projective variety. There exists a non nef extremal ray R of length $\geq (n-1)$ if and only if there exists a morphism $\varphi : X \to X'$ into a smooth projective variety X' which is the blow-up of X' at a point and whose exceptional locus is exactly the set of curves in the ray R.*

Let us point out that in the case $\mathrm{Char}(k) > 0$ the existence of a map $\varphi : X \to X'$ which contracts all (and only) the curves in an extremal ray has not been proved in general; if $\mathrm{Char}(k) = 0$ this is the so called Kawamata-Shokurov contraction theorem.
In the surface case the assumption $l(R) \geq (n-1)$ is trivial, therefore the above Theorem in the case of surfaces is a complete characterization of non nef rays; this is the Castelnuovo contraction theorem.

Proof. The if part of the theorem is trivial.

Let R be as in the theorem and V, E as above. By the inequality

$$\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V, 0 \to x) + 1 = \dim V \geq 2n-1,$$

since on the other hand $\mathrm{Locus}(V) \subseteq \mathrm{Locus}(R) \subseteq E$ and thus

$$2n-2 \geq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V, 0 \to x),$$

we have $\mathrm{Locus}(V) = \mathrm{Locus}(V, 0 \to x) = E$, $\dim V = 2n-1$ and $-K_X.C = n-1$.
Let $f' : \mathbb{P}^1 \to \Gamma \subset E$ be a curve of the family which intersects the smooth locus of E; since every element of $\mathrm{Hom}_{[f']}(\mathbb{P}^1, E)$ is also an element of $\mathrm{Hom}_{[f']}(\mathbb{P}^1, X)$ and $\mathrm{Locus}(V) = E$ we can take an irreducible component of $\mathrm{Hom}_{[f']}(\mathbb{P}^1, E)$, call it W, which is contained in V; this implies that $2n-1 = \dim V \geq \dim W$.
On the other hand, since E is a locally complete intersection, being a divisor in a smooth variety, we can apply the inequality for the dimension of the Hilbert scheme **in** E [Kol96, Theorem II.1.3] which gives $\dim W \geq -K_E.\Gamma + n - 1$; combining the two inequalities we get

$$-K_E.\Gamma \leq n.$$

Recalling that $E.R < 0$, by the adjunction formula $K_E = (K_X + E)_{|E}$, we have that $E.\Gamma = -1$ and $-K_E.\Gamma = n$.

Note that we can apply [Kol96, IV.3.13.3] to the scheme E and the family V; in fact for a generic point $x \in E$ we have seen that $\mathrm{Locus}(V, 0 \to x) = E$.
Thus we obtain that $\mathrm{Pic}(E) = \mathbb{Z}$; moreover, from what we said above, $-E_E$ is the positive generator of $\mathrm{Pic}(E)$ and $-K_E = -nE_E$.

Let $\pi : \widetilde{E} \to E$ be the normalization of E; we have that

$$-K_{\widetilde{E}} = -\pi^* K_E + (\text{conductor of } \pi)$$

where the conductor of π is an effective divisor which is zero iff π is an isomorphism ([Har77, Ex. III.7.2]). Thus

$$-K_{\widetilde{E}}(\pi^*(-E_E))^{n-2} \geq -K_E(-E_E)^{n-2} = n(-E_E)^{n-1} = n(\pi^*(-E_E))^{n-1}.$$

We lift the curves of V to $\widetilde{E}$ and we apply [KK00, Lemma 8], see 2.5, which gives $\widetilde{E} = \mathbb{P}^{n-1}$ and $\pi^*(-E_E) = \mathcal{O}_{\mathbb{P}}(1)$. Since

$$-K_{\widetilde{E}} = -\pi^* K_E + (\text{conductor of } \pi) = n\pi^*(-E_E) + (\text{conductor of } \pi)$$

the conductor of π is zero, i.e. π is an isomorphism.

To conclude the proof of the theorem we apply the following

Theorem 2.8 *(Castelnuovo) Let $E \subset X$ be a Cartier divisor in a smooth projective variety X with $E \cong \mathbb{P}^{n-1}$ and $E_E = \mathcal{O}_{\mathbb{P}}(-1)$; then there exists a morphism $\varphi : X \to X'$ into a smooth projective variety X' which is the blow-up of X' at a point such that E is the exceptional divisor.*

A straightforward application of this result is in the so called *adjunction theory*: namely if L is an ample line bundle on a a smooth n-dimensional projective variety X one can study the ampleness of the adjoint bundle $K_X + (n-1)L$: this is ample unless there exist extremal rays of length $\geq (n-1)$.
If these rays are nef they will determine the uniruled variety X; this is the case if $l(R) > (n-1)$, see [KK00], or if $\mathrm{Char}(k) = 0$ is zero, see [Fuj87]; if $l(R) = n-1$ the precise characterization of X is still open in positive characteristic.
If there are non nef rays one can apply the above theorem and the following happens:

Theorem 2.9 *Let X be a smooth n-dimensional projective variety with $n \geq 3$ and L an ample line bundle on X.*
If there is a non nef ray in the cone $\overline{NE(X)}_{(K_X+(n-1)L)\leq 0}$ then, with the only exception given by the blow-up of $\mathbb{P}^3$ in one point, $\pi : Bl_x\mathbb{P}^3 \to \mathbb{P}^3$, and $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - [\pi^{-1}(x)]$, all rays in this cone are non nef and they can be simultaneously contracted into a smooth variety X', with a morphism $\varphi : X \to X'$ expressing X as blow-up of X' at a finite set of points B.
Moreover there is an ample line bundle L' on X' such that $L \otimes ([\varphi^{-1}(B)]) = \varphi^* L'$ and $K_{X'} + (n-1)L'$ is ample.

A similar statement holds for the case $n = 2$ where extra care is needed for conic bundles. Actually the adjunction theory and the above theorem in case of surfaces were developed exactly a century ago in a paper of Castelnuovo and Enriques ([CE01], Firenze, fall 1900).
The above theorem was proved in the case $\mathrm{Char}(k) = 0$ by A. Sommese who called the map $\varphi : X \to X'$ *the first reduction of* (X, L) (see [BS95] for a more general account as well as for applications of adjunction theory).
A more general version of the theorem can be proved considering a rank $(n-1)$ ample vector bundle $\mathcal{E}$ and studying the non nef rays in the cone $\overline{NE(X)}_{(K_X+det(\mathcal{E}))\leq 0}$ (this version of the theorem if $\mathrm{Char}(k) = 0$ was proved in [**?**]).

If we restrict to the case $\mathrm{Char}(k) = 0$ something more can be said. Actually in this case another fundamental theorem holds, namely for every extremal ray R there exists a map $f = cont_R : X \longrightarrow X'$ onto a normal projective variety and with connected fiber such that if $C \subset X$ is a curve then $\varphi(C)$ is a point if and only if $[C] \in R$; this is the so called *Kawamata-Shokurov contraction theorem*.
By definition the exceptional locus E of this morphism coincide with $Locus(R) =$ $\mathrm{Locus}(V)$ and $\mathrm{Locus}(V, 0 \to x)$ is contained in a component S of the fiber $f^{-1}(f(x))$.
The inequality in 2.1 can thus be read as

$$\dim E(\varphi) + \dim S \geq \dim X + l(R) - 1.$$

Here we have the following general Conjecture (see (2.6) in [AW97])

Conjecture 2.10 Let X be a smooth n-dimensional projective variety over an algebraically closed field of characteristic zero and $R = \mathbb{R}_+[C]$ an extremal ray. If the above inequality holds for an irreducible component S then its normalization, $S' = \mathbb{P}^s$.

We have the following results, the first is a generalization of the theorem 2.4

Theorem 2.11 *X is a projective bundle (in the analytic or etale topology) over a smooth variety Y if and only if X has a ray of length r whose associated contraction has all fibers of dimension less or equal then $r-1$ and $(K_X)^n \geq (r)^n$.*

This is proved as in the proof of 3.2 and 3.3 of [AM]. Note that this argument actually proves that if the conjecture 2.3 is true then the assumption $(K_X)^n \geq$ $(r)^n$ is superflous.
In the non nef and divisorial case we have the optimal characterization:

Theorem 2.12 *Let X be a smooth n-dimensional projective variety over an algebraically closed field of characteristic zero; the two following facts are equivalent:*
1) There exists an extremal ray R such that the contraction associated to R is

divisorial and the fibers have dimension $= l(R)$.
2) There exists a morphism $\varphi : X \to X'$ *into a smooth projective variety* X' *which is the blow-up of* X' *along a smooth subvariety of codimension* $l(R) + 1$.

In the special case of characteristic zero we can further extend our classification of non nef extremal rays with high (but not maximal) length as in the following theorem.

Theorem 2.13 *Let* X *be a smooth* n*-dimensional projective variety over an algebraically closed field of characteristic zero and* $R = \mathbb{R}_+[C]$ *a non nef extremal ray. Let* $\varphi : X \to X'$ *be the birational elementary contraction associated to* R *and* $E = Exc(\varphi)$ *its exceptional locus. If* $l(R) = n - 2$ *then one of the following cases occur:*
1) $\varphi(E)$ *is a point and* $(E, -E_E) \simeq (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(2))$.
2) $\varphi(E)$ *is a point and* $(E, -E_E) \simeq (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}}(1))$, *where* $\mathbb{Q}^{n-1}$ *is a possibly singular quadric.*
3) X' *is smooth and* φ *is the blow-up along a smooth curve* $\varphi(E) \subset X'$.

3 Second Part; lifting of special rays

A possible, and classical, approach to the classification of projective varieties is an inductive one: namely one can slice the projective variety X with a (general) hyperplane section obtaining the a new variety, $Z := X \cup H$, which is of smaller dimension . By induction one can possibly say something on the structure of Z and thus the hopes is to lift this structure to a structure on X.
Let us show how is it possible to perform the inductive step in the frame of the MMP. For this let us state our assumption.
Let X be a smooth projective variety (we assume that the base field is the complex number field, but may be many results remain true in positive characteristic with not much effort) and let $Z \subset X$ be a submanifold which is a section of a vector bundle E on X of rank $r = dimX - dimZ$. Assume also that E is ample.
The first assumption implies in particular that $N_{Z/X} = E$, thus by adjunction formula that $(K_X + detE)_{|Z} = K_Z$; the second that $det(E)$ is an ample line bundle and that $det(E) \cdot C \geq r$ for every rational curve C on X.
By the assumption therefore we have that if Z is not minimal then X is not minimal; assume Z has a special ray, and thus a special associated contraction, is it possible to lift this ray, or the contraction, to X?

Example 3.1 Let $Z = \mathbb{P}^1 \times \mathbb{P}^s$, then Z can be constructed as smooth section of a rank $r = n - (s + 1)$ ample vector bundle on a manifold $p : X = \mathbb{P}(G) \longrightarrow \mathbb{P}^1$ where G is a vector bundle on $\mathbb{P}^1$ (see **??**). The restriction of p to Z is the projection to the first factor while the projection into the second factor cannot be lifted up to X.

The problem in this case is that, althought the Mori-Kleiman cone is in both cases (Z and X) of dimension two, in the first case the two extremal rays are both in the negative part of the cone while in the last one ray is in the positive part.

Remark 3.2 There is a conjecture of Sommese which says that this is essentially the unique case (a part from two trivial ones) in which a $\mathbb{P}^k$-bundle structure cannot be lifted up to X, see [BS95], section 5.5.

Example 3.3 Let $X = \mathbb{P}^1 \times \mathbb{P}^{(n-1)}$ and $L = \mathcal{O}(1,1)$; then a general section Z of L is isomorphich to $\mathbb{P}(\oplus^{(n-1)}\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and the projection on the second factor on X restrict to the map $Z \longrightarrow \mathbb{P}^{(n-1}$ which is the blow up of a point of $\mathbb{P}^{(n-1)}$. In particular both contraction of Z can be lifted but a nef ray is lifted to a non nef ray.

We proved the following theorems:

Theorem 3.4 *(Lifting of extremal faces) Assume that Z is not minimal and that dim $Z \geq 2$. Let H be an ample line bundle on X and consider the extremal face on Z, F_Z, supported by $D_Z = (K_Z + \tau H_Z)$, where τ is the nef value of H_Z (i.e. $\tau = min\{t \in \mathbb{R} : K_Z + tH_Z is\ nef\}$). Then the divisor $D = K_X + det\mathcal{E} + \tau H$ is nef but not ample and defines an extremal face F_X of X which, under the identification $N_1(Z) \simeq N_1(X)$, coincides with F_Z.*
In this situation we will say that the face F_Z is liftable *to X.*

Theorem 3.5 *We assume that, in the above notation, the face F_Z has a ray which is a scroll contraction $\varphi : Z \longrightarrow W$. Then X has a Fano-Mori contraction $\phi : X \to W$ which is of fiber type and with supporting divisor $D = K_X + det\mathcal{E} + (n - m - r + 1)H$. Moreover ϕ is elementary and it is a scroll contraction (i.e. it is supported by the divisor $K_X + (n - m + 1)H$).*

Corollary 3.6 *If a smooth projective variety X has an ample vector bundle with a section which is a $\mathbb{P}$ bundle over a minimal manifold S then X is a $\mathbb{P}$-bundle over S.*

The corollary answers positively to the Sommese's conjecture we have quoted above in the case when the base of the ¶-bundle is minimal. In particular, with other known results and some smal additional arguments, it solves the conjecture in the case $dim Z = 3$.

Theorem 3.7 *We assume that, in the above notation, the face F_Z has a ray which is the blow up $\varphi : Z \longrightarrow Z'$ of a smooth subvariety $C \subset Z'$ of codimension $m \geq 3$, then Z' is isomorphic to a subvariety of X', $\phi : X \to X'$ is the blow-up of C and $\phi_Z = \varphi$, i.e. φ is liftable to X.*

References

[ABW93] M. Andreatta, E. Ballico, and J. A. Wiśniewski. Two theorems on elementary contractions. *Math. Ann.*, **297** (1993), 191-198.

[AM] M. Andreatta, M. Mella. Contractions on a manifold polarized by an ample vector bundle. *Transactions of the AMS*, **349** (1997), 4669-4683.

[A01] M. Andreatta,G. Occhetta. Ample vector bundles with sections vanishing on special varieties. *International J. of Math.*, **10** (1999), 677-696.

[A02] ——————. Special rays in the Mori cone of a projective variety. AG/0010101 prepint.

[AW93] M. Andreatta and J. A. Wiśniewski. A note on nonvanishing and applications. *Duke Math. J.*, **72** (1993), 739–755.

[AW97] ——————. A view on contraction of higher dimensional varieties. In *Algebraic Geometry – Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 153–183. Amer. Math. Soc., Providence, RI, 1997.

[AW97] ——————. On contractions of smooth varieties. *Journal of Alg. Geo.*, **7** (1998), 253-312.

[BS95] M. C. Beltrametti and A. J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *Exp. Math.* de Gruyter, Berlin, 1995.

[CE01] G. Castelnuovo and F. Enriques. Sopra alcune questioni fondamentali nella teoria delle superficie algebriche. *Annali di matematica pura ed applicata*, **VI** (1901), 165–225.

[Fuj87] T. Fujita. On polarized manifold whose adjoint bundles are not semipositive. In *Algebraic geometry, Sendai*, volume 10 of *Adv. Studies in Pure Math.*, pages 167–178. Kinokuniya-North-Holland, 1987.

[Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *GTM*. Springer-Verlag, 1977.

[Keb00] S. Kebekus. Families of rational curves of low degree. preprint AG/0004023, April 2000.

[KK00] Y. Kachi and J. Kollár. Characterization of $\mathbb{P}^n$ in arbitrary characteristic. preprint AG/0003230, March 2000.

[Kol96] J. Kollár *Rational Curves on Algebraic Varieties*, volume 32 of *Ergebnisse der Math.* Springer Verlag, Berlin, Heidelberg, New York, Tokio, 1996.

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