Conference in Schiermonnikoog

Lifting from an ample section

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Problem: Compare the geometries of X and Y

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The general principle is that a given ample section gives strong restriction to the ambient.
Actually many manifolds cannot be ample divisors of any manifolds.

Sommese result

In the above quoted paper (1976) Sommese proved:

Theorem. (assume $r = 1$) Let $p : Y \to Z$ be a holomorphic surjection with $dimY - dimZ \geq 2$. Then there exists $\hat{p} : X \to Z$ which extends p.

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Corollary. $Y = Y^1 \times Y^2$ with $dimY^i \geq 2$ cannot be an ample divisor

Badescu example

Let $Y = \mathbb{P}^1 \times \mathbb{P}^{n-2}$. By the above result the first projection extends $p_1 : X \to \mathbb{P}^1$ and one can prove that $X = \mathbb{P}(\mathcal{G})$.

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Let

$$0 \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1} \longrightarrow \oplus^n(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(s-a)) \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1}(s) \longrightarrow 0$$

with a, s such that $0 < s - a < a$.
This gives $Y = \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset X = \mathbb{P}(\mathcal{G})$ as an ample section of $\mathcal{E} = \oplus^n \xi_{\mathcal{G}}$ with $\mathcal{G} = \oplus^n(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(s-a))$

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The p_2 fibration does not extend (otherwise we would have a surjective map $\mathbb{P}^{n-1} \to \mathbb{P}^{n-2}$).

another conjecture

Sommese conjectured that the above example is the only case of a $\mathbb{P}^k$-bundle which is a ample divisor and whose contraction does not lift to the ambient.

The conjecture is true for $k \geq 2$ and if the target Z has dimension ≤ 2 or if it is minimal (in the MMP sense).

Birational case - Fujita's theorem

Theorem. (assume $r = 1$) Let $p : Y : Bl_C Z \to Z$ be the blow-up of Z along $C \subset Z$ with $codim(C, Z) \geq 3$. Then there exists $\hat{p} : X \to \hat{Z}$ where $X = Bl_C \hat{Z}$.... .

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The bound is sharp: Let

$$0 \to \oplus^n \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a-1) \oplus^{n-3} \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1) := \mathcal{G}$$

$$\to \oplus^{n-2} \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1) \to 0$$

This gives $Y = Bl_{\mathbb{P}^{n-3}} \mathbb{P}^{n-1} \subset X = \mathbb{P}(\mathcal{G})$ as a ample section.

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The blow up contraction does not extend (Y is Fano while X is not).

Lefschetz

Weak Lefschetz: $\mathrm{N}^1(X) \to \mathrm{N}^1(Y)$ is an isomorphism if $\dim Y \geq 3$ (surjective if $\dim Y = 2$).

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$N^1(X) \supset A(X)$ ample cone, $\overline{A(X)}$ nef cone

$N_1(X) \supset NE(X)$ effective curves, $\overline{NE(X)}$ Mori-Kleiman.

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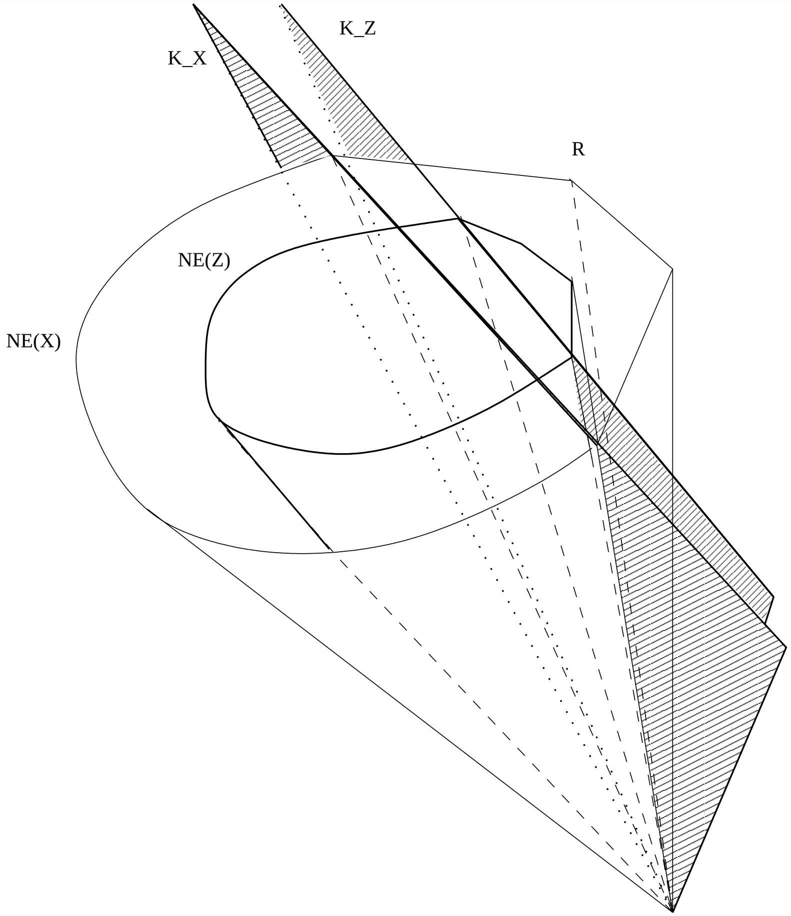
Study the polyhedral part of the cone, i.e $\overline{NE(X))}_{K_X \leq 0}$.

cone



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Let $R \in NE(X)_{K_Y \leq 0}$ ($(K_X + det\mathcal{E}) \cdot R = K_Y \cdot R \leq 0$) extremal, then it is (extremal) in Y.

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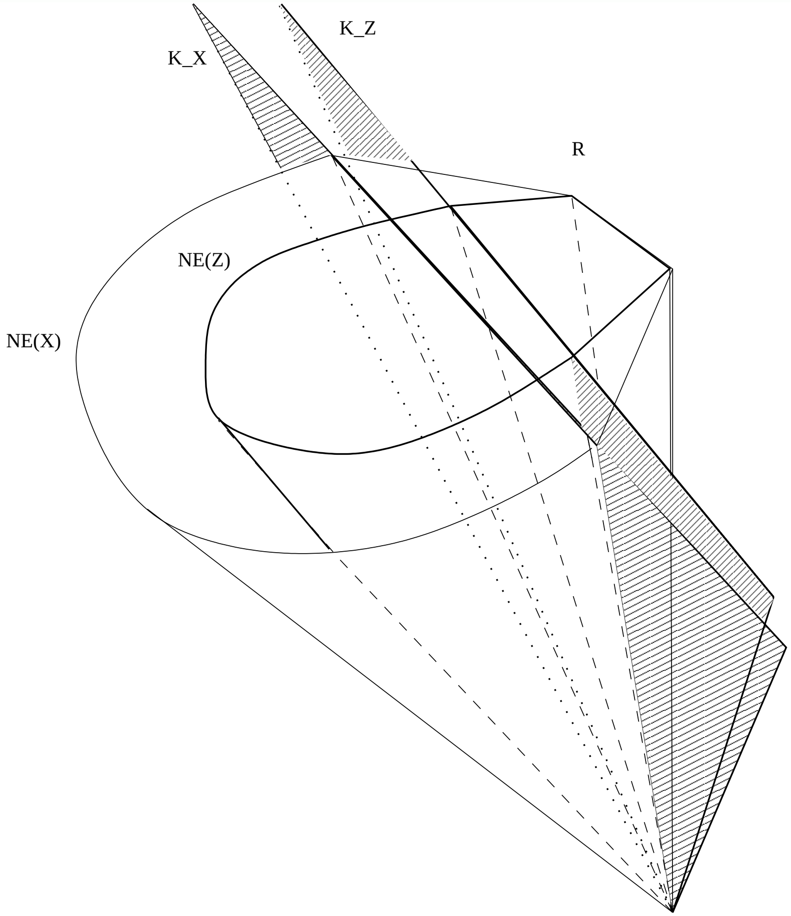
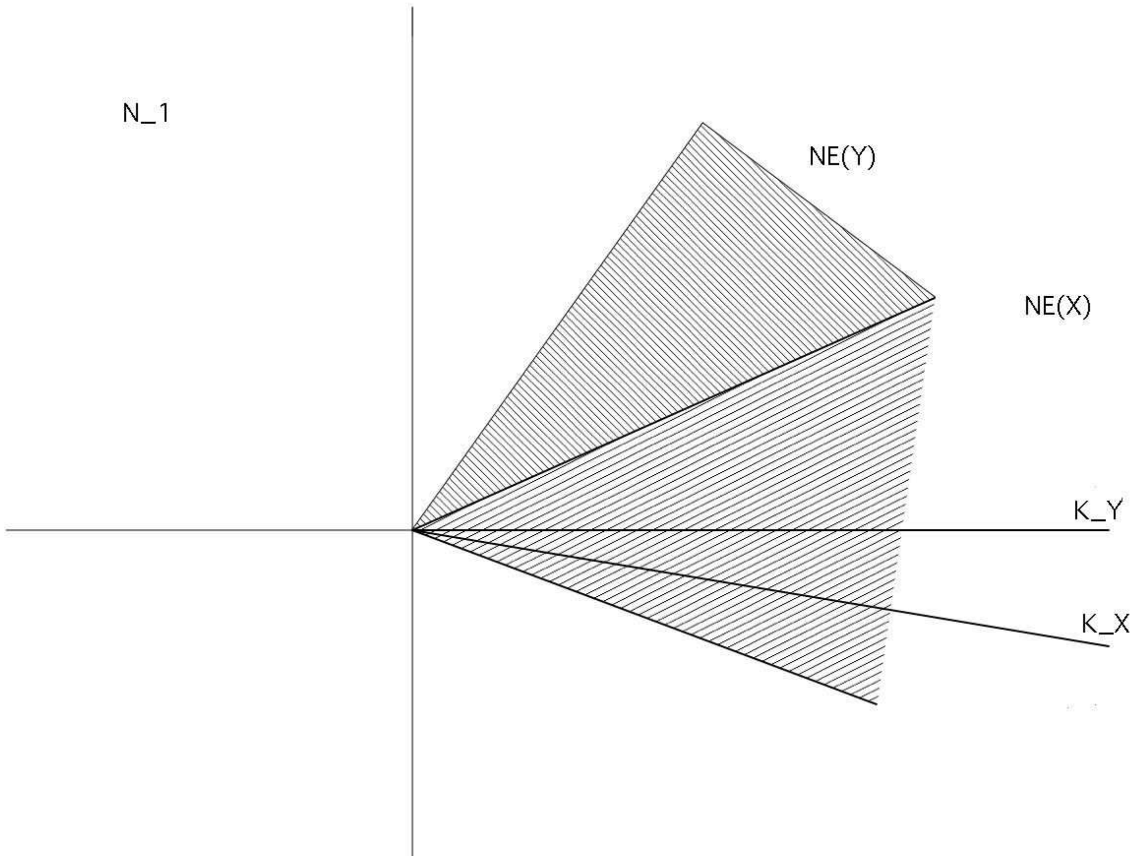
Corollary If Y is not minimal then there exists at least a common ray.
Subcorollary Sommese's conjecture is true if Z is not minimal (and if it is a surface).

cone 2



cone of the example



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Remark The two above examples are the only known of rays in $NE(Y)_{K_Y<0}$ which do not lift. (Easy to find in the positive part of the cone).

Question: If $\rho \geq 3$ are rays in $NE(Y)_{K_Y<0}$ extremal in X?

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Take a supporting divisor of R in Y of the type $K_Y + \tau L_Y$, with L_Y ample. If L_X is ample then R is extremal in X.

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Theorem (Andreatta- Occhetta) Let Y be a Fano manifold of index r, i.e. $-K_Y = rH_Y$ with H_Y spanned and $r \geq \frac{dimY}{2}$. Then X is Fano and $NE(X) = NE(Y)$, unless $Y = \mathbb{P}^1 \times V$ with $V = \mathbb{P}^3$ or a del Pezzo manifold with $\rho(V) = 1$.

Families of rational curves

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Theorem (Occhetta, Beltrametti-de Fernex- Lanteri)
Let $R = \mathbb{R}_+[C] \in \overline{NE(Y)_{K_Y<0}}$ and let $V_X \subset Hom(\mathbb{P}^1, X)$ an irreducible family containing $f : \mathbb{P}^1 \to C$.
If V_X is a covering family , this is the case if R is of fiber type and we choose C in a covering family of Y, and numerically-unsplit then R is extemal in X.

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If V_X is a covering family , this is the case if R is of fiber type and we choose C in a covering family of Y, and numerically-unsplit then R is extemal in X.

Remark The assumption about numerically unsplit is necessary. In the first example it is not attained

Proof: breaking lemma

Lemma For every curve $\Gamma \in Locus(V, \{0\} \to Y)$ we have

$$\Gamma = a\Gamma_Y + bC$$

in $N_1(X)$, where C is a curve in $\overline{V}$, Γ_Y is a curve in Y and $a \geq 0$.

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Proof

Rational connected fibrations

Given a family of rational curve V dense in X we say that $x \sim_{rc_V} y$ iff there exists a chain of curves in $\overline{V} \subset Chow_1(X)$ containing x and y

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$X \dashrightarrow X//V$ is the rc_V-fibration and
$X//V$ is a rc_V-quotient.

Lifting rc_V- fibrations, 1

Theorem (Occhetta, Beltrametti-de Fernex - Lanteri)
Let $Y \subset X$ with $N_{Y/X}$ ample and $N^1(X) \to N^1(Y)$ surjective. Let V_Y be a family of rational curve dense in Y.

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1) Its extension V_X is dense in X and, for suitable choice of the quotient, there exists a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, dashed, "\pi"'] & X \arrow[d, dashed, "\varphi"] \\ Y//V_Y \arrow[r, "\delta"] & X//V_X \end{tikzcd}$$

Where π and φ are the rc_V fibrations and δ is a surjective morphism.

Lifting rc_V- fibrations, 2

2) Moreover δ is generically finite if one of the following holds:
i) $H^0(Y//V_Y, K_{Y//V_Y}) \neq 0$
ii) $dimY - dimY//V_Y > r$
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Proposition If Y is a smooth section of $\mathcal{E}$ then the last condition is an if and only if (if one takes V_Y generically numerically unsplit). In the first example the family V_X is not generically numerically unsplit, while V_Y is unsplit.

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Question If a rc_V fibration on a manifold X is a regular morphism on a ample section Y is it regular on X? (a positive answer will solve Sommese's comjecture).

Mori Dream Spaces

Definition. A normal projective $\mathbb{Q}$-factorial variety X is a Mori Dream Space if

1) $Pic(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$ (i.e. $H^1(\mathcal{O}_X) = 0$)

2) $Cox(X, \mathbb{L}) = \oplus_{m \in \mathbb{Z}^r} H^0(X, \mathbb{L})$ is finitely generated, where $\mathbb{L} = (L_1, ..., L_r)$ is a basis for $Pic(X)_{\mathbb{Q}}$.

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Let $N^1(X, \mathbb{L}) \subset N^1(X)$ be the subgroup generated by $\mathbb{L}^m$ for $m \in \mathbb{Z}^r$ and $T_{\mathbb{L}} = Hom(N^1(X, \mathbb{L}), \mathbb{C}^*)$.
The natural grading corresponds to an action of $T_{\mathbb{L}}$ on $V = Spec(Cox(X, \mathbb{L}))$.
We can consider the GIT quotient $V//_{\chi}T$ with the trivial bundle $\mathcal{O}_V$ and a T-linearization by an ample character χ.

Mori Dream Spaces

Proposition A (Hu-Keel)
If X is a MDS then V_χ^{ss} does not depend on the choice of χ, $X = V//_\chi T$ and
i) V_χ^{un} has codimension ≥ 2 in V
ii) $V_\chi^{ss} = V_\chi^{st}$
iii) the maps $N^1(X)_{\mathbb{Q}} \to Pic^T(V_\chi^{ss})$ and $Pic(X)_{\mathbb{Q}} \to Pic^T(V_\chi^{ss})$ are isomorphisms.

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Proposition B (Hu-Keel)
Let T be a torus acting on an affine variety V and χ be a character of T. If $X = V//_\chi T$ is projective $\mathbb{Q}$-factorial and i), ii), iii) hold then X is a MDS.

MDS, examples

Examples

1. Toric varieties with $Pic(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$ are MDS (Cox: the Cox ring is a polynomial ring iff X is toric).

2. Fano manifolds are MDS (Birkahr-Cascini-Hacon-McKernan).

Ample sections of MDS

Theorem (Jow Shin-Yao)
Let X be a smooth MDS of dimension ≥ 4.
If $codim(V_X^{un}, V) \geq 3$ then every smooth ample divisor $Y \subset X$ is a MDS.
Moreover $Nef(X) = Nef(Y)$.

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Question When a MDS Y is an ample section? and the cones are equal?

Theorem (Batyrev- Mel'nikov)
A smooth toric variety Y is an ample section iff
$Y = \mathbb{P}^{n-1} \subset X = \mathbb{P}^n$ or
$Y = \mathbb{P}(\mathcal{F}) \subset X = \mathbb{P}(\mathcal{G})$ with $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{G} \to \mathcal{F} \to 0$.