The projective space

M. Andreatta

Università di Trento, Italy

www.science.unitn.it/~ andreatt

Notations

$X =$ smooth projective variety of dimension n,
$TX =$ tangent bundle and
$K_X = -det(TX)$ canonical bundle.

Problem

Find simple conditions under which the manifold X is actually the projective space $\mathbb{P}^n$.
A very famous condition is the following, conjectured by Frenkel and Hartshorne:

Theorem [Mori; 79]. *X is the projective space if and only if TX is ample.*

One of the big idea introduced by Mori is the fact of using rational curves on Fano manifolds. With this tecnique many results were proved since then.

Generalization in two directions:

Theorem [Kebekus 01, Cho- Miyaoka- Sh. Barron 01]
($k = \mathbb{C}$). X is the projective space if and only if $-K_X \cdot C \geq (n+1)$ for any curve $C \subset X$.

Theorem [Andreatta-Wiśniewski; 01]
($k = \mathbb{C}$). X is the projective space if and only if there exists an ample subsheaf $E \subset TX$ (and $E = T\mathbb{P}^n$ or $E = \oplus^r \mathcal{O}_{\mathbb{P}}(1)$).

A slight more general formulation:

Theorem *($k = \mathbb{C}$). Let $\varphi : X \to Y$ be a surjective map to a normal variety Y with $m := dimY < n$ and let $E \subset T_{X/Y}$ be a subsheaf of the relative tangent bundle.*

- If E is ample then $X = \mathbb{P}^n$.

- If E is φ-ample and φ is equidimensional and smooth then $\varphi : X \to Y$ is a scroll.

Scketch of proof.

By our assumption X **is uniruled** (Mori and Miyaoka criterium). Let $V \subset Hom(\mathbb{P}^1, X)$ be a closed irreducible component which covers X (that is $Locus(V) = X$).

By the Mori's bend and break trick we can take V to be minimal or generically unsplit; actually our assumptions actually implies that V **is unsplit, i.e. the family is proper**.

Equivalence relation: $x_1,\ x_2 \in X$ are *Rational chain connected with respect to* V , $x_1 (rcV)\, x_2$ if there exists a chain of rational curves in V which joins x_1 and x_2.
(Campana;Kollár-Miyaoka-Mori):

Theorem. *There exists an open subset* $X^0 \subset X$ *and a proper surjective morphisms with connected fibers* $\varphi^0 : X^0 \to Z^0$ *onto a normal variety and the fibers of* φ^0 *are equivalence classes of rc*V *relation.*

First case: Z_0 is just a point, X is r.c.V.

Let $V_x = V \cap Hom(\mathbb{P}^1, X; 0 \mapsto x)$ for general x .
- V_x is compact and $dim V_x = -K_X{\cdot}C - 2$.
- V_x^{sing} has dimension at most one and $V_x^{sing,x}$ is finite with at most nodes (Kebekus).
Consider a derivative map (Mori; Mok-Hwang)

$$\Phi_x : V_x \to P(T_x X) = P((f^*TX)_0)$$

$$\Phi_x([f]) = [(df)_0(\partial/\partial t)]$$

Facts
- Φ_x is a regular map (Ke) and it is finite (Ke).
- Φ_x is surjective (clear if $-K_X{\cdot}C \geq (n+1)$, some work in the case of ample subbundle, if the $rkE = 1$ we have to pass through a result of J. Wahl).
- if Φ_x is surjective then it is birational and thus a biholomorphism (by a lemma of Miyaoka which says that if Φ_x is not generically injective then there exists a curve in V singular at x).

Second case: $dim(Z_0) > 0$.

This case does not occur in the Ke, C-M-S theorem.

It brings to contradiction in the other case by proving that φ^0 can be extended in codimension 2 and that over a curve it is a $\mathbb{P}^k$-bundle.

Theorem [Kachi- Kollár, 00, Andreatta 02]. *Let E be an ample vector bundle on X, $r = rkE$ (for instance $E = \oplus^r L$, L ample line bundle) .*

i) If $r \geq n$, then $K_X + detE$ is nef unless $X = \mathbb{P}^n$ and $E = \oplus^n \mathcal{O}_{\mathbb{P}}(1)$.

ii) If $r = n-1$ then either $K_X + detE$ is nef or the pair (X, E) is one of the following:
$X = \mathbb{P}^n$ and $E = \mathcal{O}_{\mathbb{P}}(2) \oplus (\oplus^{n-2}\mathcal{O}_{\mathbb{P}}(1))$,
$X = \mathbb{P}^n$ and $E = \oplus^{n-1}\mathcal{O}_{\mathbb{P}}(1)$,
X is a quadric $\mathbb{Q}^n \subset \mathbb{P}^{(n+1)}$ and $E = \oplus^{n-1}\mathcal{O}_{\mathbb{P}}(1)$,
X is the projectivization of a rank n vector bundle over a smooth curve A, $\varphi : X \to A$, and $E = \varphi^*(\varphi_*(E(-1))) \otimes \mathcal{O}_{\mathbb{P}}(1)$.

The proof uses families V of rational curves on X; the assumptions implies the existence of a covering family of rational curve which are lines with respect to a polarization.

Let R be an extremal ray whose length is $l(R) = \min \{-K_X.C : [C] \in R\}$. A ray is non nef if $R{\cdot}D < 0$ for some effective $D \subset X$ (equiv. Locus$(R) \neq X$).

Theorem[Andreatta-Occhetta 01] (any k) *There exists a non nef extremal ray R of length $\geq (n-1)$ if and only if there exists a morphism $\varphi : X \to X'$ into a smooth projective variety X' which is the blow-up of X' at a point.*

The proof uses families V of rational curves on the exceptional locus; the assumptions implies the existence of a covering family of rational curve which are lines with respect to $-D$.

For $n = 2$ this is exactly Castelnuovo contraction theorem.

In the case $k = \mathbb{C}$ we get a similar characterization of non nef extremal rays of length $(n-2)$ (higher dimensional equivalent of Mori's contractions).

Of a rather different nature is the following characterization

Conjecture [Fujita 80?]
Let L be an ample line bundle.
$K_X + (n+1)L$ is very ample if and only if $(X, L) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1))$.

It is true, and not so difficult, if L is very ample or just ample and spanned.