# The projective space

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## Notations

$X =$ smooth projective variety of dimension $n$,
$TX =$ tangent bundle and
$K_X = -det(TX)$ canonical bundle.

## Problem

Find simple conditions under which the manifold $X$ is actually the projective space $\mathbb{P}^n$.

A very famous one is the following,
(conjectured by Frenkel and Hartshorne):

**Theorem [Mori 79].**
$X = \mathbb{P}^n$ *if and only if* $TX$ *is ample.*

To prove it Mori introduced the use of :
*rational curves on Fano manifolds*.
This method has been developed to study (uniruled) projective manifolds.

We have for example the following two generalizations of Mori's theorem (over $\mathbb{C}$):

**Theorem [Cho-Miyaoka-Shepherd Barron01] [Kebekus 01]** $(k = \mathbb{C})$

*$X = \mathbb{P}^n$ if and only if $-K_X{\cdot}C \geq (n+1)$ for any rational curve $C \subset X$.*

**Theorem[Andreatta-Wiśniewski 01]**$(k = \mathbb{C})$

*$X = \mathbb{P}^n$ if and only if there exists an ample subsheaf $E \subset TX$
(and $E = T\mathbb{P}^n$ or $E = \oplus^r \mathcal{O}_{\mathbb{P}}(1)$).*

A slight more general formulation of the last:

**Theorem** *($k = \mathbb{C}$).*
*Let $\varphi : X \to Y$ be a surjective map to a normal variety $Y$ with $m := dim Y < n$ and let $E \subset T_{X/Y}$ be a subsheaf of the relative tangent bundle.*

*- If $E$ is ample then $X = \mathbb{P}^n$.*
*- If $E$ is $\varphi$-ample and $\varphi$ is equidimensional and smooth then $\varphi : X \to Y$ is a scroll.*

## Sketch of proof.

By assumptions $X$ **is uniruled** (Mori and Miyaoka criterium).

Let $V \subset Hom(\mathbb{P}^1, X)$ be a closed irreducible component which covers $X$ ($Locus(V) = X$).

By the Mori's bend and break trick we can take $V$ to be minimal or generically unsplit; our assumptions actually implies that $V$ **is unsplit, i.e. the family is proper**.

**Equivalence relation**:
$x_1,\ x_2 \in X$ are *Rational chain connected with respect to* $V$ , $x_1 (rcV)\, x_2$ if there exists a chain of rational curves in $V$ which joins $x_1$ and $x_2$.

Campana and Kollár-Miyaoka-Mori:

**Theorem**. *There exists an open subset $X^0 \subset X$ and a proper surjective morphisms with connected fibers $\varphi^0 : X^0 \to Z^0$ onto a normal variety and the fibers of $\varphi^0$ are equivalence classes of rc$V$ relation.*

The proof splits in two cases:

**I:** $Z_0$ is just a point, $X$ is rationally connected with respect to the family $V$.

**II:** $dim(Z_0) > 0$
(this will bring to a contradiction).

**I.**

Let $V_x = V \cap Hom(\mathbb{P}^1, X; 0 \mapsto x)$ for general $x$ . $V_x^{sing,x}$ is finite with at most nodes (Kebekus).

Consider a *derivative map* (Mori; Mok-Hwang)

$$\Phi_x : Norm(V_x) \to P(T_x X) = P((f^*TX)_0)$$

$$\Phi_x([f]) = [(df)_0(\partial/\partial t)]$$

- $\Phi_x$ is a regular map and it is finite (Kebekus).
- $\Phi_x$ is surjective
(this is clear if $-K_X \cdot C \geq (n+1)$ since $V_x$ is compact and $dim V_x = -K_X \cdot C - 2$, some work in the case of ample subbundle, if the $rkE = 1$ we have to pass through a result of J. Wahl).
- if $\Phi_x$ is surjective then it is birational and thus a biholomorphism (by a lemma of Miyaoka which says that if $\Phi_x$ is not generically injective then there exists a curve in $V$ singular at $x$).
- $X$ is $\mathbb{P}^n$ (an argument in Mori).

**II:** $dim(Z_0) > 0$.

(This case does not occur in the C-M-S or Ke theorem.)

We prove that
- $\varphi^0$ can be extended in codimension 2

and subsequently that
- over a curve it is a $\mathbb{P}^k$-bundle and $E$ is a subsheaf of the relative tangent bundle;

$\mathbb{P}^k$-bundle do not have subsheaf of the relative tangent bundle which are ample, by an argument of Campana-Peternell.

In positive characteristic the above two results are not known.
Using a *polarization* (i.e. an ample vector bundle) we have :

**Theorem [Kachi- Kollár 00, Andreatta 02]**.
*Let $E$ be an ample vector bundle on $X$, $r = rkE$ (Kachi-Kollár considered the case $E = \oplus^r L$).*

*If $r \geq n$, then $K_X + detE$ is nef unless $X = \mathbb{P}^n$ and $E = \oplus^n \mathcal{O}_{\mathbb{P}}(1)$.*

*If $r = n-1$ then either $K_X + detE$ is nef or the pair $(X, E)$ is one of the following:*

*- $X = \mathbb{P}^n$ and $E = \mathcal{O}_{\mathbb{P}}(2) \oplus (\oplus^{n-2}\mathcal{O}_{\mathbb{P}}(1))$ ,*
*- $X = \mathbb{P}^n$ and $E = \oplus^{n-1}\mathcal{O}_{\mathbb{P}}(1)$,*
*- $X$ is a quadric $\mathbb{Q}^n \subset \mathbb{P}^{(n+1)}$ and $E = \oplus^{n-1}\mathcal{O}_{\mathbb{P}}(1)$,*
*- $X$ is the projectivization of a rank $n$ vector bundle over a smooth curve $A$, $\varphi : X \to A$, and $E = \varphi^*(\varphi_*(E(-1))) \otimes \mathcal{O}_{\mathbb{P}}(1)$.*

(In the case $k = \mathbb{C}$ this was known since a while: Sommese, Fujita, Peternell, Ye-Zhang,..)

The assumptions implies the existence of a covering family of rational curve which are *lines* with respect to a polarization:

(($r \geq n$) Assume ($K_X + detE$) not nef:
then we have an extremal ray $R = \mathbb{R}^+[C]$ and, since $detE{\cdot}C \geq r$ for every rational curve,

$$-K_X{\cdot}C = (n+1)$$

$$detE{\cdot}C = r = n,$$

$$(-K_X + detE){\cdot}C = 1.)$$

**Lemma [Kachi- Kollár]**. *Let $\{C_t\}$ be a family of curves through $x \in X$ which covers X. Assume that $(C_t.H) = 1$ and $(-K_X.H^{n-1}) > (n-1)(H^n)$ for an ample $H$.*
*Then $X \cong \mathbb{P}^n$, $H = \mathcal{O}_{\mathbb{P}^n}(1)$ .*

*Maps whose fibers are $\mathbb{P}^r$:* **the blow-up**.

**Terminology:**
$R$: *extremal ray* in the Mori cone.
$l(R) :=$ min $\{-K_X.C : [C] \in R\}$ : *length of* $R$.
$R$ is *non nef* if $R{\cdot}D < 0$ for some effective divisor $D \subset X$ (equivalently Locus($R$) $\neq X$).

**Theorem [Andreatta-Occhetta 01]** (any $k$)
*$X$ projective manifold of dimension $n$. It has a non nef extremal ray $R$ with $l(R) \geq (n-1)$ if and only if it has a morphism $\varphi : X \to X'$ into a smooth projective variety $X'$ which is the blow-up of $X'$ at a point.*

For $n = 2$ this is exactly Castelnuovo contraction theorem.

In the case $k = \mathbb{C}$ we get a similar characterization of non nef extremal rays of length $\geq (n-2)$ (higher dimensional equivalent of Mori's contractions).

The proof uses families $V$ of rational curves on the *exceptional locus*.

The assumptions implies the existence of a rational curve $C$ in the ray with $-K_X \cdot C = n - 1$, which is contained in a family $V$ such that $Locus(V) = Locus(V_x) = D$, a divisor.

Extimating the dimension of the deformation of $C$ *within* $D$ we obtain that

$$n \geq -K_D \cdot C = -K_X \cdot C + D \cdot C$$

i.e. $-D \cdot C = 1$; we can apply the above Lemma.

Unsettled characterizations of $\mathbb{P}^n$ (or rather good properties of projective varieties).

**Conjecture [Fujita 80?]**
Let $X$ be a projective manifold of dimension $n$ and $L$ be an ample line bundle.
$(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1))$ if and only if $K_X + (n+1)L$ is not very ample.
(Easy if $L$ is very ample.)

**Conjecture [Hwang 01?]**
$\mathbb{P}^n$ is, between the Fano manifold with $Pic = \mathbb{Z}$, the one with largest possible automorphism group.