Varietá simplettiche olomorfe costruzioni classiche e calcolo degli invarianti

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Introduction

Theorem of Berger: the holonomy group of a connected Riemanian manifold (M, g), not symmetric, irreducible and simply connected, is one of the following

$SO(n)$,
$U(m) \subset SO(2m)$,
$SU(m) \subset SO(2m)$, $m \geq 3$,
$Sp(r) \subset SO(4r)$,
$Sp(1)Sp(r) \subset SO(4r)$, $r \geq 2$,
$G_2 \subset SO(7)$
$Spin(7) \subset SO(8)$

Holonomy characterization

If we fix the holonomy group we impose certain parallel tensor fields; smaller is the holonomy more special is the manifold.

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$H \subset U(m) \subset SO(2m)$:
iff H commutes with endomorphism $v \to iv$ of $\mathbb{R}^{2m} = \mathbb{C}^m$
iff parallel endomorphism J of $T(M)$, $J^2 = -I$
iff M has a Kähler complex structure J.

Calabi Yau

$H \subset SU(m)$:
iff $H \subset U(m)$ and H preserves the $\mathbb{C}$-multilinear alternating m-form $det : \mathbb{C}^m \to C$
iff M is Kähler and there exists an holomorphic parallel m-form $\omega \neq 0$

The viceversa follows from Yau's theorem.

Hyperkähler-Symplectic manifolds

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$Sp(r)$ is the subgroup of $O(\mathbb{R}^{4r})$ commuting with i, j, k.

If $H \subset Sp(r)$ then M has parallel complex structures I, J, K and it is called hyperkähler.

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"Cayley": $\mathbb{C} = \mathbb{R} + \mathbb{R}i$, $\mathbb{H} = \mathbb{C}(j)$ with $jz = \overline{z}j$; $\mathbb{H} \equiv \mathbb{C}^{2r}$.
$\psi = h + \varphi j$: h is $\mathbb{C}$-hermitian and φ is $\mathbb{C}$ bilinear alternating.
Thus $Sp(r) = U(2r, \mathbb{C}) \cap Sp(2r, \mathbb{C})$ and
if $H \subset Sp(r)$ then M has a complex kähler structure + a parallel holomorphic symplectic 2-form φ, unique up to a scalar.

Bogomolov decomposition theorem

Theorem Let X be a compact Kähler manifold with $c_1(X) = 0$. Then X has a finite unramified cover Y such that

$$Y = Z \times \Pi S_i \times \Pi C_j$$

where

- Z is a a complex torus
- each S_i is a simply connected holomorphic symplectic manifold with $H^2(S_i, \mathcal{O}_{S_i}) = 1$
- each C_i is a simply connected Calabi Yau manifold with $H^2(C_i, \mathcal{O}_{C_i}) = 0$

Example of Symplectic-Hyperkähler

$r = 1$, $Sp(1) = SU(2)$:
M is a compact complex surface with a non zero holomorphic 2-form and $\pi_1 = 0$ (assume only $b_1 = 0$).
They are the so called $K3$-surfaces.
For instance a quartic in $\mathbb{P}^3$. They are all deformations of each others, thus all diffeomorphic and with $\pi_1 = 0$.

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Kummer surfaces Take an abelian surface A and consider the action of $\mathbb{Z}_2$ on A by involution $a \to -a$. The quotient has 16 simple double points. In particular it admits a crepant resolution $X \to Y/\mathbb{Z}_2$, X is a $K3$-surface (called Kummer).

(Generalized) Kummer construction

- Take an integral (irreducible) representation of a finite group $\rho_{\mathbb{Z}} : G \to GL(r, \mathbb{Z})$.
- Take an abelian variety A of dimension d and extend ρ to $\rho_A = \rho_{\mathbb{Z}} \otimes_{\mathbb{Z}} A : G \to Aut(A^r)$.
- If d not even then assume $\rho_{\mathbb{Z}} : G \to SL(r, \mathbb{Z})$.
- The representation on TA^r and $H^1(A^r, \mathbb{C})$ is $d \cdot \rho_{\mathbb{C}}$.
- Take the quotient $Y = A^r/G$, find a crepant resolution $X \to Y$, get a complex manifold with $H^1(X, \mathbb{C}) = 0$ and $K_X = 0$, i.e. a Calabi-Yau or a Symplectic manifold.

More general construction

More generally:

- Consider a finite group of automorphisms of an abelian variety A, i.e. $G < Aut(A)$
- The tangent action at the unit of A is a complex representation of G, that is $\rho : G \to GL(TA)$.
- The same representation is in cohomology $\rho : G \to GL(H^1(A, \mathbb{C}))$.
- Want trivial invariant subspace and $\rho(G) < SL(H^1(A, \mathbb{C}))$

Problem: existence of a crepant resolution

- On dimension 2 and 3 we know a lot about crepant resolutions but this is not the case in higher dimensions.
- For solvable groups we can take towers of resolutions of abelian singularities, provided at each step we get an equivariant one.
- Via Hilbert schemes: for a smooth surface S the Hilbert scheme $Hilb^n(S)$ provides a crepant resolution $Hilb^n(S) \to Sym^n(S)$ (classical, Fogarty). .
  For a curve, C, $Sym^n(C)$ is already smooth; in higher dimension it is not true.

Other $K3$'s

Finite subgroups of $SL(2,\mathbb{Z})$, up to conjugation in $GL(2,\mathbb{Z})$, are $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ generated respectively by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

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One can classify finite subgroups of $SL(2,\mathbb{C})$ (besides the cyclic groups the dihedral groups, the binary tetrahedral, octahedral and icosahedral groups); locally they give du Val singularities. Therefore if the more general construction apply we get other $K3$ surfaces.

CY 3*-folds*

The following are, up to isomorphism, (non-trivial) finite subgroups of $SL(3,\mathbb{Z})$:

- cyclic groups $\mathbb{Z}_a$, of rank a, for $a = 2,\ 3,\ 4$ and 6,
- dihedral groups D_{2a}, of rank $2a$, for $a = 2,\ 3,\ 4$ and 6, which have, respectively, $4,\ 3,\ 2$ and 1 conjugacy classes in $GL(3,\mathbb{Z})$
- the alternating group A_4 which has 3 conjugacy classes in $GL(3,\mathbb{Z})$ (e.g. the tetrahedral group of isometries of the tetrahedron),
- the symmetric group S_4 which has 3 conjugacy classes in $GL(3,\mathbb{Z})$ (e.g. octahedral group of isometries of a cube)

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Some of this groups which are not conjugate in $GL(3, \mathbb{Z})$ are in $GL(3, \mathbb{C})$; they give non-isomorphic CY 3- folds.

Symplectic quotient

Let $G < Sp(2n, \mathbb{C})$ be a finite subgroup which acts on $\mathbb{C}^{2n}$ preserving a symplectic form.
The quotient $\mathbb{C}^{2n}/G$ is a symplectic variety, i.e. the smooth part admits a holomorphic symplectic form σ such that its pull back to any resolution extends to a holomorphic 2-form (Beauville).

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A proper morphism $\varphi : X \to V/G$ is a symplectic resolution if X is smooth and $\varphi^*(\hat{\sigma})$ extends to a symplectic form on X.

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A resolution in this case is symplectic if and only if it is crepant.

Necessary conditions for the existence of symplectic resolution

Theorem. If a symplectic resolution $\varphi : X \to V/G$ exists then G is generated by symplectic reflections, i.e. elements g such that $codim V^g = 2$.

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Notation: if equality holds Z is called a maximal cycle.

Examples

For example if $\rho_{\mathbb{C}} : G \to GL(V) = GL(n, \mathbb{C})$ is a representation of a finite group then $\rho \oplus \rho^* : G \to Sp(V \oplus V^*)$: the symplectic form preserved is the identity in $V \otimes V^*$.

If moreover $\rho_{\mathbb{C}}$ preserves a non degenerate symmetric 2-form on V then there is a G-equivariant isomorphism $V \simeq V^*$; this is the case when G is a Weyl group acting on the lattice of roots of a simple Lie algebra: it preserves the Killing form.

Examples

- $G = S_{n+1}$ and $\rho_{\mathbb{Z}} : S_{n+1} \to GL(n, \mathbb{Z})$ be the standard representation, i.e. the natural representation on $\mathbb{Z}^{(n+1)}$ restricted to the invariant subspace $e_0 + \cdots + e_r = 0$.
- $G_{n,m} = \mathbb{Z}_m^n \rtimes S_n$ and $\rho_{\mathbb{C}} : G_{n,m} \to GL(n, \mathbb{C})$ be the natural representation, where $\mathbb{Z}_m^r$ acts on $\mathbb{C}^r$ diagonally and S_n by permutations of the coordinates. It is an integral representation if $m = 2$.
- $G = Q_8 \rtimes \mathbb{Z}_3$, the binary tetrahedral group, and representations $\rho_{\mathbb{C}} : G \to GL(2, \mathbb{C})$:
  ρ_0 the standard arising from the embedding
  $G \subset SU(2)$, and $\rho_j := \rho_0 \otimes \mathbb{C}_j$, for $j = 1, 2$, where $\mathbb{C}_j$ is the multiplication by a third root of unity. The last two are dual to each other.

Existence of symplectic resolution

Theorem. Let G be a finite group, $\rho_{\mathbb{C}} : G \to GL(\mathbb{C}^n)$ an irreducible complex representation. Then $V \oplus V^*/G$ has a symplectic resolution if and only if $(G, \rho_{\mathbb{C}})$ is as above.

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A resolution in the first two cases can be obtained via the Hilbert scheme construction. In the third case an explicit resolution was constructed recently (Lehn-Sorger)

Kummer constr. for the examples

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constructed long ago by Beauville and Fujiki.
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The third example cannot work...

non existence result

An important tool:

Theorem (Lefschetz) Let $g : A \to A$ be an endomorphism with $g(0) = 0$ and let $\eta_(g)$ be its tangent. The closed analytic subvariety of A consisting of the fixed point of g, denoted by $Fix(g)$, has dimension equal to the multiplicity of 1 as an eigenvalue of $\eta_(g)$. If it is zero dimensional then $|Fix(g)| = |det(1 - \eta_(g))|^2$.

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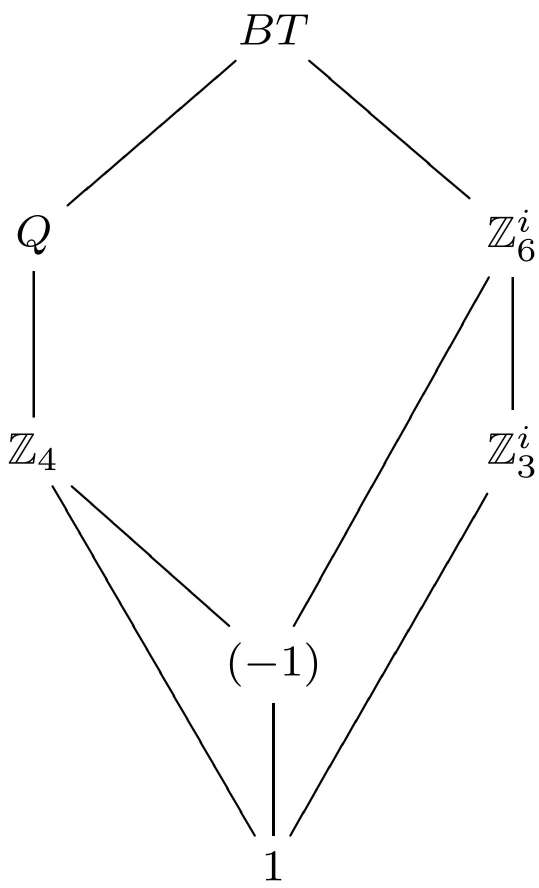
We use also semismallness of symplectic resolution: in particular if dim ≥ 4 there are no isolated quotient symplectic singularities.

non existence result

Lattice of subgroups



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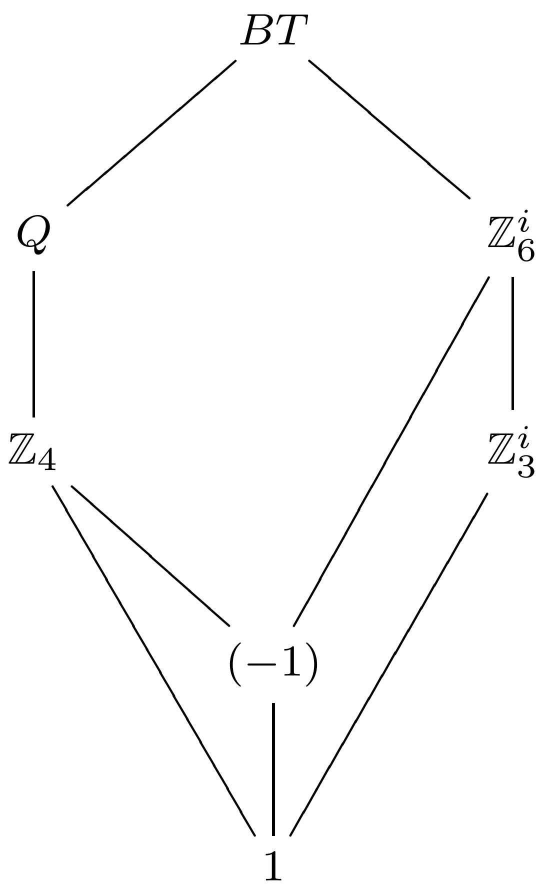

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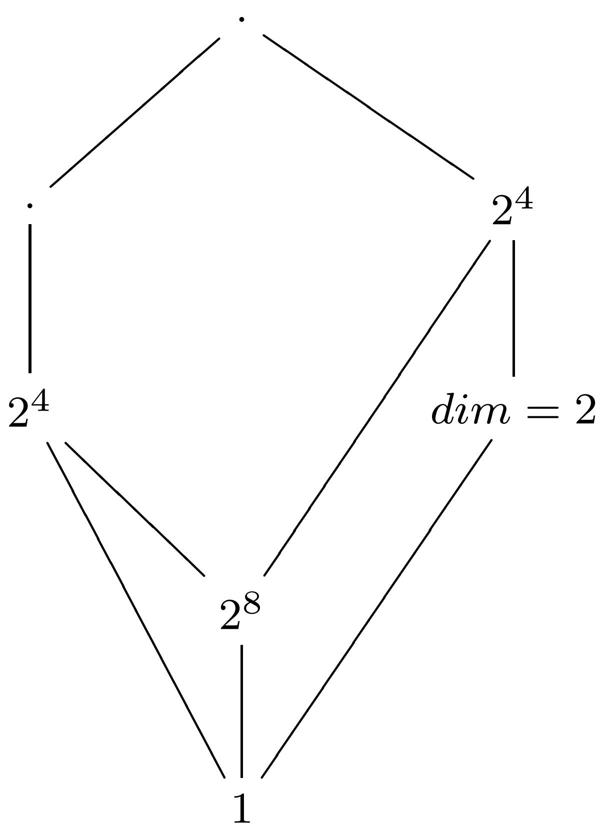

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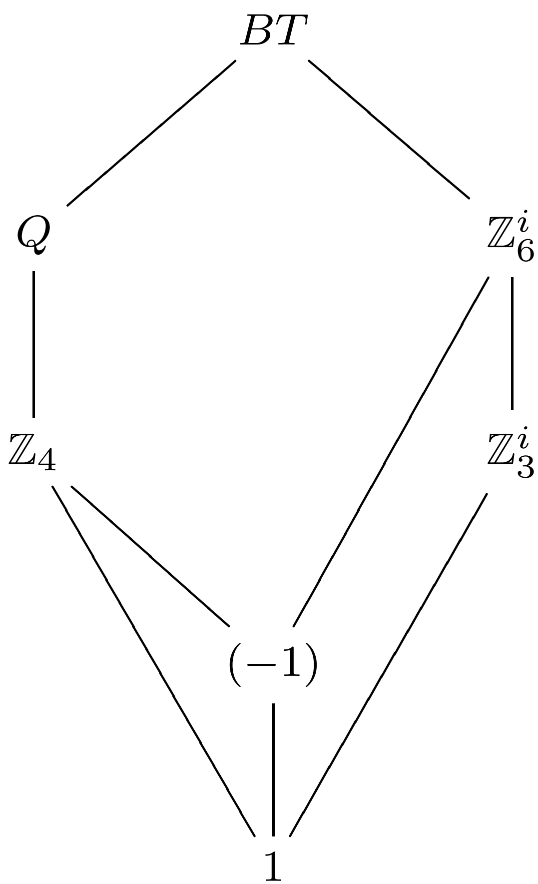

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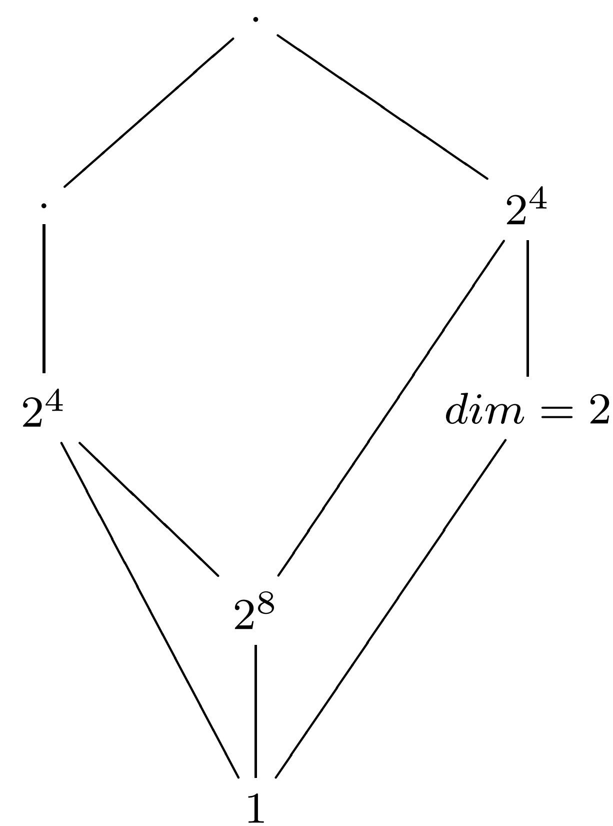

$$Fix(-1) = \cup_j (Fix(\mathbb{Z}_6^i) - Fix(BT)) \cup Fix(BT), i.e. 2^8 = 4(2^4) - 3s$$

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= Principle

$$\begin{array}{ccc} & & A \\ & & \downarrow \pi \\ X & \xrightarrow{\varphi} & Y = A/G \end{array}$$

The answer to any well posed question about the geometry of X is the G-equivariant geometry of A.

Virtual Poincaré polynomial

$P_X(t)$ virtual Poincaré polynomial is defined by:

- $P_X(t) = \sum_{i=0}^{2n} b_i(X)\, t^i \in \mathbb{Z}[t]$,
  if X is compact manifold, $n = dim X$, t is a formal variable and $b_i(X) = dim H^i_{DR}(X)$ are the Betti numbers.
- If Y is a closed algebraic subset of X and $U := X \setminus Y$ then
$$P_X(t) = P_Y(t) + P_U(t).$$

Remark that the virtual Poincaré is actually the standard Poincaré polynomial also if X is compact and has quotient singularities

G- **_Poincaré polynomial_**

Consider ring $R(G)$ of complex representations of G;
by $d\cdot\rho$, $\rho \otimes m$ and $\rho \wedge m$ we denote the sum of d copies, the m-th tensor and alternating power of ρ.
We have a map $\mu_0 : R(G) \to Z$ which to a representation ρ assigns the rank of its maximal trivial subrepresentation.

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Given action of G on variety Z define G-Poincaré polynomial $P_{Z,G}(t) \in R(G)[t]$ whose coefficient at t_i is the vector space $H^i(Z, C)$ with the induced G-action.

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In our set-up $P_{A^r,G}(t) = \sum_{i=0}^{2rd} (2d \cdot \rho_{\mathbb{C}})^{\wedge i} \cdot t^i$
For $Y = A^r/\rho_A$ we have $P_Y(t) = \mu_0(P_{A^r,G}(t))$

McKay correspondence

McKay conjecture: Let $G < SL(V)$ be a finite subgroup and assume that there exists a crepant resolution $X \to V/G$. Then the homology $H_*(X, \mathbb{Q})$ admits a "natural" basis numbered by conjugacy classes of elements $g \in G$.

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In the case $G < Sp(V)$ it has been proved by Kaledin that maximal cycles (i.e. $2\, codim(Z) = codim \varphi(Z)$) fom a basis.

Idea of computation

Strata

$Y([H]) \subset Y$: orbits of points whose isotropy is in the conjugacy class of a subgroup $H < G$.
$X([H])$ the inverse image.
The restriction $X([H]) \to Y([H])$ is a locally trivial fiber bdl with fiber $F([H])$ which embeds in the following diagram ($W(H) = N(H)/H$ is the Weil group and $(A^r)_0^H$ are the set of point whose stabilizer is H).

$$\begin{tikzcd}
\left(\overline{(A^r)_0^H} \times F([H])\right)/W([H]) \arrow[d] & X([H]) \arrow[l] \arrow[d] \\
\widehat{Y([H])} & Y([H]) \arrow[l]
\end{tikzcd}$$

Idea of computation

Poincaré of the strata (Let A_K be a component of $(A^r)_0^H$)

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P_{A_K,W_K} is obtained computing the cohomology of A_K invariant via W_K:
$P_{A_K,W_K}(t) = \sum_{i=0}^{2dr_0} (2d \cdot \eta_K)^{\wedge i} \cdot t^i = (1+t)^{2d\eta_K}$ where $\eta_K : W_K \to GL(r_K, \mathbb{C})$ is a representation of W_K induced from $\rho_{\mathbb{C}}$.

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By McKay the group $W(H)$ acts on the cohomology of $F(H)$ as $W(H)$ acts on the conjugacy classes of H.
So $P_{F(H),W_K}$ is determined by the adjoint action of W_K on conjugacy classes of elements in H, which is $w([h]_H) \mapsto [whw^{-1}]_H$.

Idea of computation

The virtual Poincaré polynomial of $X([H])$ is obtained taking out the contribution of the lower dimensional strata over the difference $\widehat{Y([H])} \setminus Y([H])$. Take therefore $H' > H$ a subgroup .... .

Cohomology of a $K3$ surface

Let A be a one dimensional torus and

$\rho(\mathbb{Z}_6) =< \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} >\subset SL(2,\mathbb{Z})$.

In $SL(2,\mathbb{C})$ $\rho = \epsilon_6 + \epsilon_6^5$, $\epsilon_6 =$ sixth primitive root of unity.

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We add to it the contribution of cohomology coming from resolving singular points of the quotient.

Cohomology of a $K3$ *surface*

g	$\#$ fix pts	$\#$ sing pts	resolution	Poincaré
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	1	1	●–●–●–●–●	$1+5t$
$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	9	4	●–●	$1+2t$
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	16	5	●	$1+t$

Cohomology of a $K3$ surface

g	# fix pts	# sing pts	resolution	Poincaré
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	1	1	●–●–●–●–●	$1+5t$
$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	9	4	●–●	$1+2t$
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	16	5	●	$1+t$

The dimension of H^{11} for a K3 surface is:

$$2 + 1 \times 5 + 4 \times 2 + 5 \times 1 = 20$$

Cohomology of a CY mfd

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A an elliptic curve.

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g	$Fix(g)$	# cmpnts	$\langle g \rangle$	$W(g)$	$\widehat{Y(\langle g \rangle)}$	Poincaré
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$2e_1 = 0$ $2e_2 = 0$	16	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$6 \times \mathbb{P}^1 1$	$1 + t^2$
$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$e_1 = e_2$ $2e_1 = 0$	4	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$4 \times \mathbb{P}^1$	$1 + (2 + \epsilon)$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$e_1 = e_2$ $2e_3 = 0$	4	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$4 \times \mathbb{P}^1$	$1 + t^2$
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$e_1 = e_2$ $e_1 = e_3$	1	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$1 \times \mathbb{P}^1$	$1 + (1 + \epsilon)$

Cohomology of a CY mfd

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subgroup	fixed set in $\{2p = 0\}$	# fixed pts	# sing pts	Poincaré
D_4	$e_1 \neq e_2 \neq e_3 \neq e_1$	24	4	$1 + 3t^2$
$3 \times D_8$	$e_i = e_j \neq e_k$, $\{i, j, k\} = \{1, 2, 3\}$	36	12	$1 + 4t^2$
$G = S_4$	$e_1 = e_2 = e_3$	4	4	$1 + 4t^2$

Cohomology of a CY mfd

3-dimensional stratum

$S3(t) := 1 + t^2 + 4t^3 + t^4 + t^6 - (15(1 + t^2 - 4) + 20)$

1-dimensional strata

$S12(t) := 10((1 + t^2)(1 + t^2) - 4(1 + t^2))$

$S13(t) := ((t^4 + 2t^3 + 2t^2 + 1) - 4(1 + t^2))$

$S14(t) := 4((2t^4 + 2t^3 + 3t^2 + 1) - 4(1 + 2t^2))$

0-dimensional stratum

$S0(t) := 4(1 + 3t^2) + (12 + 4)(1 + 4t^2)$

Calculating the sum

`P(t):=S3(t)+S12(t)+S13(t)+S14(t)+S0(t)` we get:

$$P_X(t) = t^6 + 20\,t^4 + 14\,t^3 + 20\,t^2 + 1.$$

Cohomology of $Kum^n(A)$**,** $Hilb^n(K3)$

Let A be a two dimensional torus.

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The Poincaré polynomial of the Beauville's generalized Kummer variety, i.e. a crepant resolution of A^n/S_{n+1} is :

$n = 2$:
$t^8 + 7t^6 + 8t^5 + 108t^4 + 8t^3 + 7t^2 + 1$
$n = 3$:
$t^{12}+7\,t^{10}+8\,t^9+51\,t^8+56\,t^7+458\,t^6+56\,t^5+51\,t^4+8\,t^3+7\,t^2+1$
...

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...

The Poincaré polynomial of a crepant resolution of $A^n/\mathbb{Z}_2^n \rtimes S_n$ is :

$n = 2$:
$t^8 + 23t^6 + 276t^4 + 23t^2 + 1$
...

Local symplectic resolutions

Let $\pi : X \to Y$ be a proper symplectic elementary contraction with X smooth and $dimX = 4$.

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If $Y = \mathbb{C}^4/G$ with $G < Sp(4)$ a finite subgroup we know that a (hilb type) symplectic resolution $X \to Y$ exists if $G = D_6 := \mathbb{Z}_3 \rtimes \mathbb{Z}_2 = \sigma_3$ or if $G = (\Gamma)^{\times 2} \rtimes \mathbb{Z}_2$ where $\Gamma < SL(2)$.
The resolution is elementary in the first case and in the second when $\Gamma = 1$.

Local symplectic resolutions

Therefore we know 3 proper symplectic elementary contractions $X \to Y$ with X smooth and $dim X = 4$, namely:

1) the (unique) small symplectic contraction
2) the (unique) resolution of $\mathbb{C}^4/\sigma_3$
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Conjecture Are they the only ones?

Local symplectic resolutions

Theorem A proper symplectic elementary contraction $\pi : X \to Y$ with X smooth and $dim X = 4$ is a Mori Dream Space (that is any movable divisor can be made nef and semiample after a finite number of SQM (small quasifactorial modification)).

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In fact it holds
i) Cone and contraction theorems (Mori-Kawamata)
ii) Existence of Flops (and of SQM) (Wi-Wi)
iii) Termination of Flops (Matsuki)

Example

Let $\pi : X \to Y$ be the Hilb type symplectic resolution of $Y = \mathbb{C}^4/((\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2)$.

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... or the resolution of $Y = \mathbb{C}^4/(\sigma_3 \rtimes \mathbb{Z}_3)$