Fano manifolds

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Fano manifolds are the building blocks of the MMP and they are uniruled, i.e. covered by rational curves.
Let $X$ be a Fano manifold. We define the **index**:

$$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some divisor } L\},$$
Numerical invariants

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Remark 1) $i_X = a r_X$, with $a$ a positive integer.

2) $r_X \leq i_X \leq (n + 1)$ the last inequality was proved by Mori.

Moreover $r_X = n + 1$ iff $X = \mathbb{P}^n$, by Kobayashi-Ochiai and $i_X = n + 1$ iff $X = \mathbb{P}^n$, by Cho-Miyaoka-Sh-Barron.
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3) The right invariant is the pseudodex $i_X$.

Note in fact that $X = \mathbb{P}^n \times \mathbb{P}^{n+1}$ has $r_X = 1$ and $i_X = n + 1$. 
The Picard number

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If $X$ is Fano then $NE(X)$ is polyhedral and (if $\rho \geq 2$) it "reflects" the geometry of the Fano manifold (Mori).
Conjecture of Mukai (1988):

$$\rho_X (r_X - 1) \leq n.$$ 

later generalized

$$\rho_X (i_X - 1) \leq n \quad \text{with } = \iff X \simeq (\mathbb{P}^{i_X - 1})^{\rho_X}.$$
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G.C. holds for toric varieties
Steps toward the conjecture

(a) \( n = 5 \),
(b) if \( i_X \geq \frac{n+3}{3} \) and there exists a family of rational curves \( V \) which is unsplit and covers \( X \).

The family exists if \( X \) has a fiber type contraction or it does not have small contractions.
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More generally one can prove that G.C. holds if \( i_X \geq \frac{n+k}{k} \) and there exists \((k - 2)\) families of rational curves \( V \) which are unsplit and cover \( X \).
Let us define a family of rational curves to be an irreducible component

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Rational curves

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Deformation theory + Riemann-Roch give a bound to the dimension from below: let $f : \mathbb{P}^1 \to C$ be a curve in $V$

$$\dim V \geq -K_X \cdot C + (n - 3),$$

$$\dim V_x \geq -K_X \cdot C - 2.$$
Special rational curves

Families which are minimal or almost lines:

- minimal with respect to the intersection with $-K_X$
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Remark. If $V$ is gen unsplit then:

$$dim Locus(V_x) = dim V_x + 1 \geq -K_X \cdot C - 1.$$
Rationally connected fibrations.

Let $X$ be uniruled, $x, y \in X$ and define:
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**Theorem** [Campana] and [Kollár-Miyaoka-Mori ] (1992)
The exists an open set $X^0 \subset X$ and a map $\phi^0 : X^0 \to Z^0$
which is proper, with connected fiber and whose fibers are
equivalence classes for the equivalence relation $\sim$
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One can also define:

$x \sim_{rcV} y$ iff $\exists$ a chain of rat. curves $\in V$ through $x$ and $y$.

If $V$ is unsplit the above theorem holds with $\sim_{rcV}$. 
An observation of Wisniewski

Proposition Let $V$ be an unsplit family. Then $\rho(Locus(V_x)) = 1$. 
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Lemma. Let $V$ be an unsplit family and $Y \subset X$ a closed subset such that $[V]$ does not belong to $NE(Y)$. Then

$$\dim Locus(V)_Y \geq \dim Y + \deg_{-K_X} V - 1.$$
Lemma. Let $V$ be an unsplit family and $Y \subset X$ a closed subset such that $[V]$ does not belong to $NE(Y)$. Then

$$\dim Locus(V)_Y \geq \dim Y + \deg_{-K_X} V - 1.$$ 

Proof. Let $U_Y$ be the universal family of curves in $V$ meeting $Y$; i.e. $e(U_Y) = Locus(V)_Y$ ($e$ evaluation map).

$$\dim U_Y \geq \dim Y + \deg_{-K_X} V - 1$$

Thus we have to prove that $e : U_Y \to X$ is generically finite.
Proof by drawing

Proof that $e : U_Y \rightarrow X$ is generically finite by contradiction.
If there exist $V_1, \ldots, V_k$ unsplit families of r.c. whose classes are linearly independent in $N_1(X)$ and such that $\text{Locus}(V_1, \ldots, V_k)_x \neq \emptyset$ then

$$n \geq \dim \text{Locus}(V_1, \ldots, V_k)_x \geq \sum_j (\deg V_j - 1) \geq k(i_X - 1),$$

this is simply an inductive form of the above proposition.
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For $k = \rho$ we would have the first part of the conjecture.
If $i_X > \frac{n+2}{2}$ then $\rho_X(i_X - 1) \leq n$ is equivalent to $\rho_X = 1$. 

*Hints of proof*
If $i_X > \frac{n+2}{2}$ then $\rho_X(i_X - 1) \leq n$ is equivalent to $\rho_X = 1$.

Assume by contradiction that $\rho_X > 1$. Let $V_1$ be a family which covers $X$ with $\deg_{-K_X} V_1 \leq (n + 1)$ (Mori); by assumption, $V_1$ is unsplit.
If \( i_X > \frac{n+2}{2} \) then \( \rho_X(i_X - 1) \leq n \) is equivalent to \( \rho_X = 1 \).

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Since \( \rho_X > 1 \) there must be another family \( V_2 \) whose curves are independent (cone theorem) and therefore we are in the ideal situation.
In general one try to start with an unsplit dominant family $V$ and construct the $rcV$-fibration. If the dimension of the target is zero (i.e. $X$ is rationally $V$-connected) then $\rho = 1$. 
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If not there exists a locally unsplit family $V'$ which is transverse and dominant with respect to the $rcV$-fibration (extension of Mori theorem by Kollár-Miyaoka-Mori). If we assume that $i_X > \frac{n+3}{3}$, also this family is unsplit.
In general one try to start with an unsplit dominant family $V$ and construct the $rcV$-fibration. If the dimension of the target is zero (i.e. $X$ is rationally $V$-connected) then $\rho = 1$.

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Construct the $rc(V, V')$-fibration. If the dimension of the target is zero then $\rho = 2$.

....
The second part of the conjecture

If the ideal situation is reached and we get equality then we have $V_1, \ldots, V_\rho$ families of rational curves which are unsplit, dominant, independent in $N_1(X)$ and whose sum of degree minus $\rho$ is equal to $\dim X$. 
The second part of the conjecture

If the ideal situation is reached and we get equality then we have $V_1, \ldots, V_\rho$ families of rational curves which are unsplit, dominant, independent in $N_1(X)$ and whose sum of degree minus $\rho$ is equal to $\dim X$.

A result of [Cho-Miyaoka-Sh.Barron] - [Kebekus] in the case $\rho = 1$ says that $X = \mathbb{P}^n$; building from it G. Occhetta proved that in general $X$ is the product of $\rho$ projective spaces.
Choose a ray $R$

Let $R$ be an extremal ray of $NE(X)$, let us define the length and the Locus:

$$l(R) := \min \{ m \in \mathbb{N} \mid -K_X \cdot C = m, C \in R \text{ rational curve} \}.$$ $$Locus(R) := \text{set of points on curves } C \subset R$$
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Theorem [Andreatta-Occhetta (2005)]. Let $X$ be a Fano manifold with $\rho_X \geq 2$ and let $R$ be an extremal ray.

$$l(R) + i_X \leq \dim Locus(R) + 2.$$
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Note that if $\rho = 2$ this is an improved Mukai inequality:

$$2i_X \leq l(R) + i_X \leq \dim Locus(R) + 2 \leq n + 2.$$
Equality

If equality holds and $R$ is not small then

$$X \simeq \mathbb{P}^k \times \mathbb{P}^{n-k} \text{ or } X \simeq Bl_{\mathbb{P}^k}(\mathbb{P}^n) \text{ with } k \leq \frac{n-3}{2}$$
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If equality holds for $r_X$, i.e. $l(R) + r_X = \dim Locus(R) + 2$
then $X = \mathbb{P}_{\mathbb{P}^k}(\mathcal{O}^{\oplus e-k+1} \oplus \mathcal{O}(1)^{\oplus n-e})$, where $e$ is the dimension of $Locus(R)$ and $k = n - r_X + 1$. 
Let $X$ be the blow up of a manifold $Y$ along $T \subset Y$, and let $i_X \geq \dim T + 1$ (i.e. $l(R) + i_X \geq \dim \text{Locus}(R) + 1$). Then $X$ is one of the following

1. $Bl_p(\mathbb{P}^n)$.
2. $Bl_p(\mathbb{Q}^n)$.
3. $Bl_p(V_d)$ where $V_d$ is $Bl_Y(\mathbb{P}^n)$ and $Y$ is a submanifold of dimension $n - 2$ and degree $\leq n$ contained in an hyperplane.
4. The blow up of $\mathbb{P}^n$ along a $\mathbb{P}^k$ with $k \leq \frac{n}{2} - 1$.
5. $\mathbb{P}^1 \times Bl_p(\mathbb{P}^{n-1})$.
6. The blow up of $\mathbb{Q}^n$ along a $\mathbb{P}^k$ with $k \leq \frac{n}{2} - 1$.
7. The blow up of $\mathbb{Q}^n$ along a $\mathbb{Q}^k$ with $k \leq \frac{n}{2} - 1$. 

$\rho_X \geq 2$, the blow-ups
Concerning more specifically the classification of Fano manifolds: they are classified up to dimension 3 and in higher dimension up to the index $n - 2$. 
Theorem [Chierici-Occhetta (2005)].
Let $X$ be a Fano manifold with $i_X \geq \text{dim}X - 3$; assume $\text{dim}X \geq 5$ and $\rho_X \geq 2$.
All possible cones $NE(X)$ are listed for such $X$ (in particular they are generated by $\rho_X$ rays).
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$X$ is the blow up of $\mathbb{P}^5$ along one of the following surfaces: a smooth quadric, a cubic scroll in $\mathbb{P}^4$, a Veronese surface.
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