Fano manifolds

Marco Andreatta

Dipartimento di Matematica

Trento

General frame

Let X be a (complex) compact manifold of dimension n and TX its tangent bundle. Define $-K_X = detTX$.

General frame

Let X be a (complex) compact manifold of dimension n and TX its tangent bundle. Define $-K_X = detTX$.

X is said to be a Fano manifold if $-K_X$ is ample.

General frame

Let X be a (complex) compact manifold of dimension n and TX its tangent bundle. Define $-K_X = detTX$.

X is said to be a Fano manifold if $-K_X$ is ample.

Example. TX is ample (iff X is $\mathbb{P}^n$).

General frame

Let X be a (complex) compact manifold of dimension n and TX its tangent bundle. Define $-K_X = detTX$.

X is said to be a Fano manifold if $-K_X$ is ample.

Example. TX is ample (iff X is $\mathbb{P}^n$).

Conjecture. X is uniruled (equivalently TX is not generically seminegative)

iff

X is birational to a fibrations of Fano varieties.

General frame

Let X be a (complex) compact manifold of dimension n and TX its tangent bundle. Define $-K_X = detTX$.

X is said to be a Fano manifold if $-K_X$ is ample.

Example. TX is ample (iff X is $\mathbb{P}^n$).

Conjecture. X is uniruled (equivalently TX is not generically seminegative)

iff

X is birational to a fibrations of Fano varieties.

The if part was proved by Kollár-Miyaoka-Mori,
the only if part follows from the Minimal Model conjecture.

General frame

Let X be a (complex) compact manifold of dimension n and TX its tangent bundle. Define $-K_X = detTX$.

X is said to be a Fano manifold if $-K_X$ is ample.

Example. TX is ample (iff X is $\mathbb{P}^n$).

Conjecture. X is uniruled (equivalently TX is not generically seminegative)

iff

X is birational to a fibrations of Fano varieties.

The if part was proved by Kollár-Miyaoka-Mori, the only if part follows from the Minimal Model conjecture.

Fano manifolds are the building blocks of the MMP and they are uniruled, i.e. covered by rational curves.

Numerical invariants

Let X be a Fano manifold. We define

the index:
$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some divisor} L\}$,

Numerical invariants

Let X be a Fano manifold. We define

the index:
$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some divisor} L\}$,

and the pseudoindex:
$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \subset X \text{ rational curve } \}$.

Numerical invariants

Let X be a Fano manifold. We define

the index:
$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some divisor} L\}$,

and the pseudoindex:
$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \subset X \text{ rational curve }\}$.

Remark 1) $i_X = ar_X$, with a a positive integer.

Numerical invariants

Let X be a Fano manifold. We define

the index:
$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some divisor} L\},$

and the pseudoindex:
$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \subset X \text{ rational curve }\}.$

Remark 1) $i_X = ar_X$, with a a positive integer.

2) $r_X \leq i_X \leq (n+1)$ the last inequality was proved by Mori. Moreover $r_X = n+1$ iff $X = \mathbb{P}^n$, by Kobayashi-Ochiai and $i_X = n+1$ iff $X = \mathbb{P}^n$, by Cho-Miyaoka-Sh-Barron.

Numerical invariants

Let X be a Fano manifold. We define

the index:
$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some divisor} L\}$,

and the pseudoindex:
$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \subset X \text{ rational curve } \}$.

Remark 1) $i_X = ar_X$, with a a positive integer.

2) $r_X \leq i_X \leq (n+1)$ the last inequality was proved by Mori.
Moreover $r_X = n+1$ iff $X = \mathbb{P}^n$, by Kobayashi-Ochiai and $i_X = n+1$ iff $X = \mathbb{P}^n$, by Cho-Miyaoka-Sh-Barron.

3) The right invariant is the pseudondex i_X.
Note in fact that $X = \mathbb{P}^n \times \mathbb{P}^{n+1}$ has $r_X = 1$ and $i_X = n+1$.

The Picard number

For a projective variety X we denote, as usual, by $N_1(X)$ the vector space generated by irreducible complex curves modulo numerical equivalence and by $\rho(X)$ its dimension.

The Picard number

For a projective variety X we denote, as usual, by $N_1(X)$ the vector space generated by irreducible complex curves modulo numerical equivalence and by $\rho(X)$ its dimension.

The cone of effective cycles, the so called Mori-Kleimann cone, will be denoted by $NE(X) \subset N_1(X)$

The Picard number

For a projective variety X we denote, as usual, by $N_1(X)$ the vector space generated by irreducible complex curves modulo numerical equivalence and by $\rho(X)$ its dimension.

The cone of effective cycles, the so called Mori-Kleimann cone, will be denoted by $NE(X) \subset N_1(X)$

If X is Fano then $NE(X)$ is polyhedral and (if $\rho \geq 2$) it "reflects" the geometry of the Fano manifold (Mori).

A conjecture of **Mukai**

Conjecture of Mukai (1988):

$$\rho_X(r_X - 1) \leq n.$$

later **generalized**

$$\rho_X(i_X - 1) \leq n \text{ with} = \text{iff} \quad X \simeq (\mathbb{P}^{i_X - 1})^{\rho_X}.$$

Steps toward the conjecture

-(1990) Wiśniewski:
If $i_X > \frac{n+2}{2}$ then $\rho_X = 1$.

Steps toward the conjecture

-(1990) Wiśniewski:
If $i_X > \frac{n+2}{2}$ then $\rho_X = 1$.

-(2001) Cho-Miyaoka-Shepherd.Barron:
G.C. holds if $i_X \geq n+1$

Steps toward the conjecture

-(1990) Wiśniewski:
If $i_X > \frac{n+2}{2}$ then $\rho_X = 1$.

-(2001) Cho-Miyaoka-Shepherd.Barron:
G.C. holds if $i_X \geq n+1$

-(2002) Bonavero, Casagrande, Debarre e Druel:
G.C. holds if (a) $n = 4$,
(b) X is toric and $i_X \geq \frac{n+3}{3}$ or $n \leq 7$.

Steps toward the conjecture

-(1990) Wiśniewski:
If $i_X > \frac{n+2}{2}$ then $\rho_X = 1$.

-(2001) Cho-Miyaoka-Shepherd.Barron:
G.C. holds if $i_X \geq n+1$

-(2002) Bonavero, Casagrande, Debarre e Druel:
G.C. holds if (a) $n = 4$,
(b) X is toric and $i_X \geq \frac{n+3}{3}$ or $n \leq 7$.

-(2004) Casagrande:
G.C. holds for toric varieties

Steps toward the conjecture

-(2004) Andreatta,Chierici,Occhetta: G.C. holds if
(a) $n = 5$,
(b) if $i_X \geq \frac{n+3}{3}$ and there exists a family of rational curves V which is unsplit and covers X.

The family exists if X has a fiber type contraction or it does not have small contractions.

Steps toward the conjecture

-(2004) Andreatta,Chierici,Occhetta: G.C. holds if
(a) $n = 5$,
(b) if $i_X \geq \frac{n+3}{3}$ and there exists a family of rational curves V which is unsplit and covers X.

The family exists if X has a fiber type contraction or it does not have small contractions.

Unfortunately there are Fano manifolds with no such a family (for which G.C. of course holds).

Steps toward the conjecture

-(2004) Andreatta,Chierici,Occhetta: G.C. holds if
(a) $n = 5$,
(b) if $i_X \geq \frac{n+3}{3}$ and there exists a family of rational curves V which is unsplit and covers X.

The family exists if X has a fiber type contraction or it does not have small contractions.

Unfortunately there are Fano manifolds with no such a family (for which G.C. of course holds).

More generally one can prove that G.C. holds if $i_X \geq \frac{n+k}{k}$ and there exists $(k-2)$ families of rational curves V which are unsplit and cover X.

Rational curves

Let us define a family of rational curves to be an irreducible component

$$V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)/\mathrm{Aut}(\mathbb{P}^1)$$

Rational curves

Let us define a family of rational curves to be an irreducible component

$$V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)/\mathrm{Aut}(\mathbb{P}^1)$$

(V_x is the subfamily of the curves passing through x).

Rational curves

Let us define a family of rational curves to be an irreducible component

$$V \subset \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)/\mathrm{Aut}(\mathbb{P}^1)$$

(V_x is the subfamily of the curves passing through x).

Deformation theory+Rieman-Roch give a bound to the dimension from below: let $f : \mathbb{P}^1 \to C$ be a curve in V

$$dimV \geq -K_X{\cdot}C + (n-3),$$

$$dimV_x \geq -K_X{\cdot}C - 2.$$

Special rational curves

Families which are `minimal` or almost lines:

- minimal with respect to the intersection with $-K_X$

Special rational curves

Families which are `minimal` or almost lines:

- minimal with respect to the intersection with $-K_X$
- unsplit if V is proper.

Special rational curves

Families which are `minimal` or almost lines:

- minimal with respect to the intersection with $-K_X$
- unsplit if V is proper.
- locally unsplit if V_x is proper.

Special rational curves

Families which are `minimal` or almost lines:

- minimal with respect to the intersection with $-K_X$
- unsplit if V is proper.
- locally unsplit if V_x is proper.
- generically unsplit if through two generic points pass only finitely many curves in the family.

Special rational curves

Families which are `minimal` or almost lines:

- minimal with respect to the intersection with $-K_X$
- unsplit if V is proper.
- locally unsplit if V_x is proper.
- generically unsplit if through two generic points pass only finitely many curves in the family.

(minimal $\Rightarrow$ unsplit $\Rightarrow$ locally unsplit $\Rightarrow$ gen. unsplit).

Special rational curves

Families which are `minimal` or almost lines:

- minimal with respect to the intersection with $-K_X$
- unsplit if V is proper.
- locally unsplit if V_x is proper.
- generically unsplit if through two generic points pass only finitely many curves in the family.

(minimal $\Rightarrow$ unsplit $\Rightarrow$ locally unsplit $\Rightarrow$ gen. unsplit).

Remark. If V is gen unsplit then:

$$dimLocus(V_x) = dimV_x + 1 \geq -K_X{\cdot}C - 1.$$

Rationally connected fibrations.

Let X be uniruled, $x, y \in X$ and define :
$x \sim y$ iff $\exists$ a chain of rational curves through x and y.

Rationally connected fibrations.

Let$X$ be uniruled, $x, y \in X$ and define :
$x \sim y$ iff $\exists$ a chain of rational curves through x and y.

Theorem [Campana] and [Kollár-Miyaoka-Mori] (1992)
The exists an open set $X^0 \subset X$ and a map $\varphi^0 : X^0 \to Z^0$ which is proper, with connected fiber and whose fibers are equivalence classes for the equivalence relation $\sim$ (fibers are rationally connected).

Rationally connected fibrations.

Let$X$ be uniruled, $x, y \in X$ and define :
$x \sim y$ iff $\exists$ a chain of rational curves through x and y.

Theorem [Campana] and [Kollár-Miyaoka-Mori] (1992)
The exists an open set $X^0 \subset X$ and a map $\varphi^0 : X^0 \to Z^0$ which is proper, with connected fiber and whose fibers are equivalence classes for the equivalence relation $\sim$ (fibers are rationally connected).

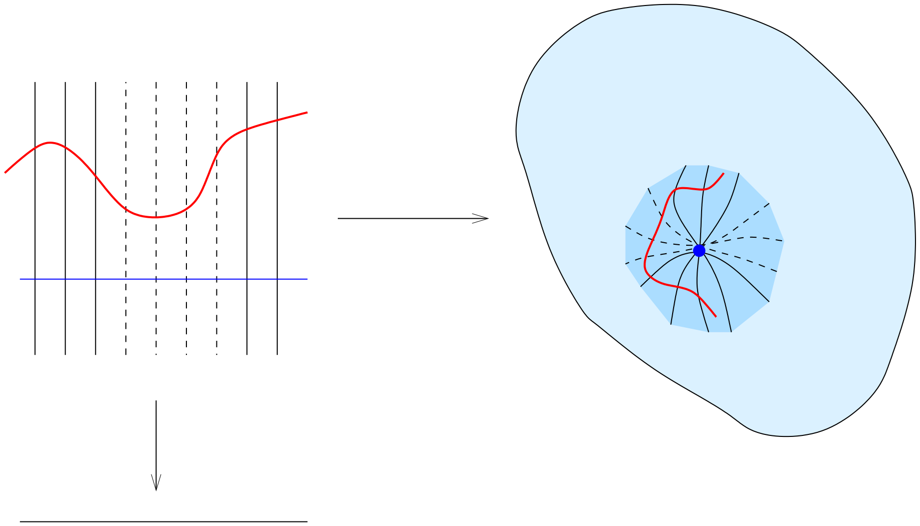
One can also define:
$x \sim_{rcV} y$ iff $\exists$ a chain of rat. curves $\in V$ through x and y.
If V is unsplit the above theorem holds with $\sim_{rcV}$.

An observation of Wisniewski

Proposition Let V be an unsplit family.
Then $\rho(Locus(V_x)) = 1$.

An observation of Wisniewski

Proposition Let V be an unsplit family.
Then $\rho(Locus(V_x)) = 1$.



the lemma

Lemma. Let V be an unsplit family and $Y \subset X$ a closed subset such that $[V]$ does not belong to $NE(Y)$. Then

$$dimLocus(V)_Y \geq dimY + deg_{-K_X}V - 1.$$

Lemma. Let V be an unsplit family and $Y \subset X$ a closed subset such that $[V]$ does not belong to $NE(Y)$. Then

$$dim Locus(V)_Y \geq dim Y + deg_{-K_X} V - 1.$$

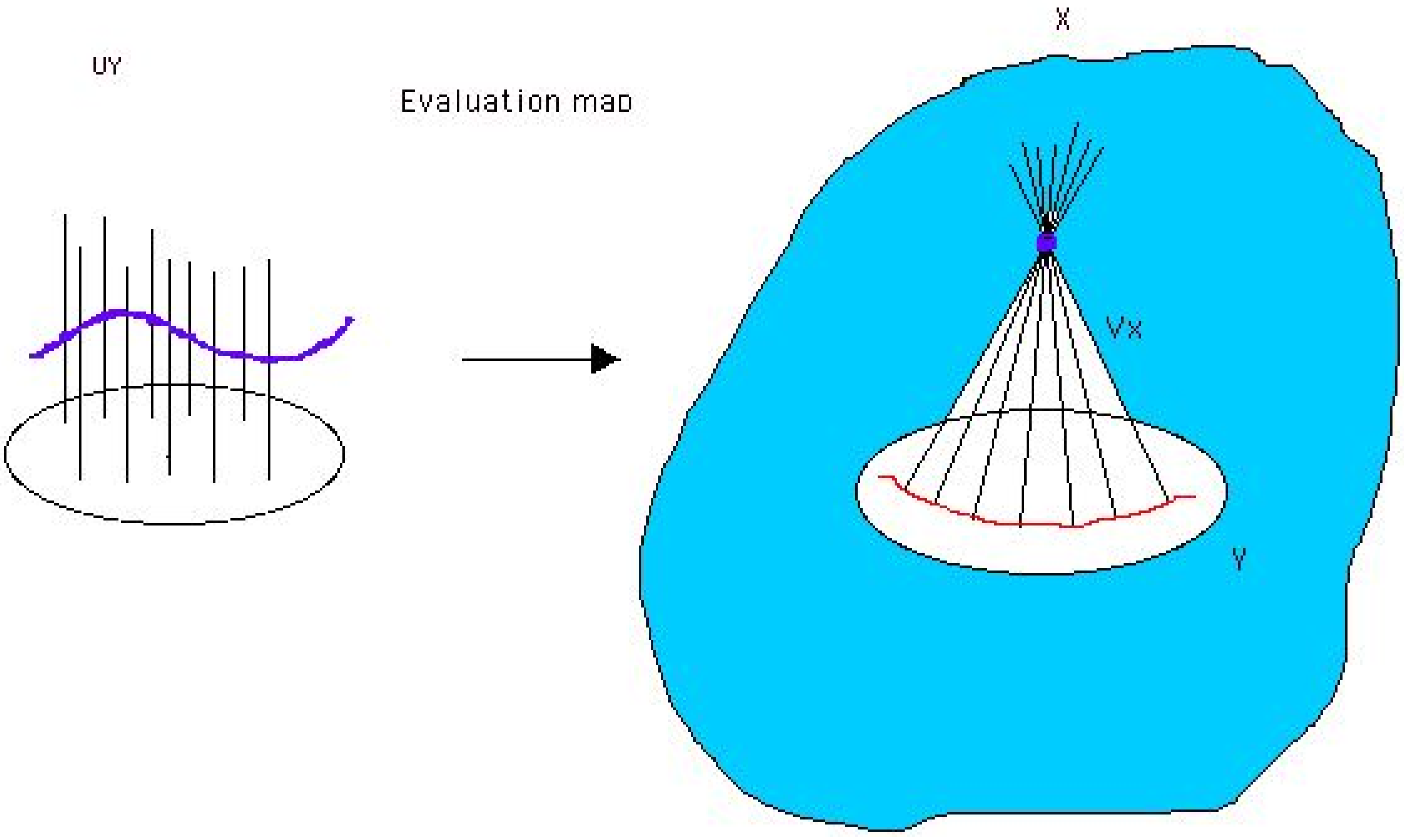
Proof. Let U_Y be the universal family of curves in V meeting Y; i.e. $= e(U_Y) = Locus(V)_Y$ (e evaluation map).

$dim U_Y \geq \dim Y + deg_{-K_X} V - 1$

Thus we have to prove that $e : U_Y \to X$ is generically finite.

proof by drawing

Proof that $e : U_Y \to X$ is generically finite by contradiction.



Ideal situation

If there exist $V_1, ..., V_k$ unsplit families of r.c. whose classes are linearly independent in $N_1(X)$ and such that $Locus(V_1, ..., V_k)_x \neq \emptyset$ then

$$n \geq dimLocus(V_1, ..., V_k)_x \geq \Sigma_j(degV_j - 1) \geq k(i_X - 1),$$

this is simply an inductive form of the above proposition.

Ideal situation

If there exist $V_1, ..., V_k$ unsplit families of r.c. whose classes are linearly independent in $N_1(X)$ and such that $Locus(V_1, ..., V_k)_x \neq \emptyset$ then

$$n \geq dimLocus(V_1, ..., V_k)_x \geq \Sigma_j(degV_j - 1) \geq k(i_X - 1),$$

this is simply an inductive form of the above proposition.

For $k = \rho$ we would have the first part of the conjecture.

hints of proof

If $i_X > \frac{n+2}{2}$ then $\rho_X(i_X - 1) \leq n$ is equivalent to $\rho_X = 1$.

hints of proof

If $i_X > \frac{n+2}{2}$ then $\rho_X(i_X - 1) \leq n$ is equivalent to $\rho_X = 1$.

Assume by contradiction that $\rho_X > 1$. Let V_1 be a family which covers X with $deg_{-K_X} V_1 \leq (n+1)$ (Mori);
by assumption, V_1 is unsplit.

hints of proof

If $i_X > \frac{n+2}{2}$ then $\rho_X(i_X - 1) \leq n$ is equivalent to $\rho_X = 1$.

Assume by contradiction that $\rho_X > 1$. Let V_1 be a family which covers X with $deg_{-K_X}V_1 \leq (n+1)$ (Mori); by assumption, V_1 is unsplit.

Since $\rho_X > 1$ there must be another family V_2 whose curves are independent (cone theorem) and therefore we are in the ideal situation.

hints of proof

In general one try to start with an unsplit dominant family V and construct the rcV-fibration.
If the dimension of the target is zero (i.e. X is rationally V-connected) then $\rho = 1$.

hints of proof

In general one try to start with an unsplit dominant family V and construct the rcV-fibration.
If the dimension of the target is zero (i.e. X is rationally V-connected) then $\rho = 1$.

If not there exists a locally unsplit family V' which is transverse and dominant with respect to the rcV-fibration (extension of Mori theorem by Kollár-Miyaoka-Mori).
If we assume that $i_X > \frac{n+3}{3}$, also this family is unsplit.

hints of proof

In general one try to start with an unsplit dominant family V and construct the rcV-fibration.
If the dimension of the target is zero (i.e. X is rationally V-connected) then $\rho = 1$.

If not there exists a locally unsplit family V' which is transverse and dominant with respect to the rcV-fibration (extension of Mori theorem by Kollár-Miyaoka-Mori).
If we assume that $i_X > \frac{n+3}{3}$, also this family is unsplit.

Construct the $rc(V, V')$-fibration. If the dimension of the target is zero then $\rho = 2$.

....

The second part of the conjecture

If the ideal sitution is reached and we get equality then we have $V_1, ..., V_\rho$ families of rational curves which are unsplit, dominant, independent in $N_1(X)$ and whose sum of degree minus ρ is equal to $dimX$.

The second part of the conjecture

If the ideal sitution is reached and we get equality then we have $V_1, ..., V_\rho$ families of rational curves which are unsplit, dominant, independent in $N_1(X)$ and whose sum of degree minus ρ is equal to $dim X$.

A result of [Cho-Miyaoka-Sh.Barron] - [Kebekus] in the case $\rho = 1$ says that $X = \mathbb{P}^n$;
building from it G. Occhetta proved that in general X is the product of ρ projective spaces.

Choose a ray R

Let R be an extremal ray of $NE(X)$, let us define the length and the Locus:

$$l(R) := \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \in R \text{ rational curve }\}.$$

$$Locus(R) := \text{ set of points on curves } \; C \subset R$$

Choose a ray R

Let R be an extremal ray of $NE(X)$, let us define the length and the Locus:

$$l(R) := \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \in R \text{ rational curve } \}.$$

$$Locus(R) := \text{ set of points on curves } \; C \subset R$$

Theorem [Andreatta-Occhetta (2005)]. Let X be a Fano manifold with $\rho_X \geq 2$ and let R be an extremal ray.

$$l(R) + i_X \leq \dim Locus(R) + 2.$$

Choose a ray R

Let R be an extremal ray of $NE(X)$, let us define the length and the Locus:

$$l(R) := \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \in R \text{ rational curve }\}.$$

$$Locus(R) := \text{ set of points on curves } \; C \subset R$$

Theorem [Andreatta-Occhetta (2005)]. Let X be a Fano manifold with $\rho_X \geq 2$ and let R be an extremal ray.

$$l(R) + i_X \leq \dim Locus(R) + 2.$$

Note that if $\rho = 2$ this is an improved Mukai inequality:

$$2i_X \leq l(R) + i_X \leq \dim Locus(R) + 2 \leq n + 2$$

Equality

If equality holds and R is not small then
$X \simeq \mathbb{P}^k \times \mathbb{P}^{n-k}$ or $X \simeq Bl_{\mathbb{P}^k}(\mathbb{P}^n)$ with $k \leq \frac{n-3}{2}$

Equality

If equality holds and R is not small then $X \simeq \mathbb{P}^k \times \mathbb{P}^{n-k}$ or $X \simeq Bl_{\mathbb{P}^k}(\mathbb{P}^n)$ with $k \leq \frac{n-3}{2}$

If equality holds for r_X, i.e. $l(R) + r_X = \dim Locus(R) + 2$ then $X = \mathbb{P}_{\mathbb{P}^k}(\mathcal{O}^{\oplus e-k+1} \oplus \mathcal{O}(1)^{\oplus n-e})$, where e is the dimension of $Locus(R)$ and $k = n - r_X + 1$.

$\rho_X \geq 2$, *the blow-ups*

Let X be the the blow up of a manifold Y along $T \subset Y$, and let $i_X \geq \dim T + 1$ (i.e. $l(R) + i_X \geq \dim Locus(R) + 1$). Then X is one of the following

1. $Bl_p(\mathbb{P}^n)$.
2. $Bl_p(\mathbb{Q}^n)$.
3. $Bl_p(V_d)$ where V_d is $Bl_Y(\mathbb{P}^n)$ and Y is a submanifold of dimension $n-2$ and degree $\leq n$ contained in an hyperplane.
4. The blow up of $\mathbb{P}^n$ along a $\mathbb{P}^k$ with $k \leq \frac{n}{2} - 1$.
5. $\mathbb{P}^1 \times Bl_p(\mathbb{P}^{n-1})$.
6. The blow up of $\mathbb{Q}^n$ along a $\mathbb{P}^k$ with $k \leq \frac{n}{2} - 1$.
7. The blow up of $\mathbb{Q}^n$ along a $\mathbb{Q}^k$ with $k \leq \frac{n}{2} - 1$.

Classification

Concerning more specifically the classification of Fano manifolds:
they are classified up to dimension 3 and in higher dimension up to the index $n-2$.

high pseudoindex, but $\rho_X \geq 2$

Theorem [Chierici-Occhetta (2005)].
Let X be a Fano manifold with $i_X \geq dimX - 3$;
assume $dimX \geq 5$ and $\rho_X \geq 2$.
All possible cones $NE(X)$ are listed for such X
(In particular they are generated by ρ_X rays).

high pseudoindex, but $\rho_X \geq 2$

Theorem [Chierici-Occhetta (2005)].
Let X be a Fano manifold with $i_X \geq dimX - 3$;
assume $dimX \geq 5$ and $\rho_X \geq 2$.
All possible cones $NE(X)$ are listed for such X
(In particular they are generated by ρ_X rays).

X has an elementary fiber type contraction except when:
X is the blow up of $\mathbb{P}^5$ along one of the following surfaces:
a smooth quadric, a cubic scroll in $\mathbb{P}^4$, a Veronese surface.

high pseudoindex, but $\rho_X \geq 2$

Theorem [Chierici-Occhetta (2005)].
Let X be a Fano manifold with $i_X \geq dimX - 3$;
assume $dimX \geq 5$ and $\rho_X \geq 2$.
All possible cones $NE(X)$ are listed for such X
(In particular they are generated by ρ_X rays).

X has an elementary fiber type contraction except when:
X is the blow up of $\mathbb{P}^5$ along one of the following surfaces:
a smooth quadric, a cubic scroll in $\mathbb{P}^4$, a Veronese surface.

For many of these cones all possible X are listed.....