The projective space and other simple varieties.

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In this paper I would like to present some recent results on the problem of finding conditions under which a smooth projective variety X defined over an algebraically closed field k is actually the projective space $\mathbb{P}^n$.

In the 1979 S. Mori proved the following celebrated theorem, which was previously conjectured by Frenkel and Hartshorne.

Theorem [17]. *$X = \mathbb{P}^n$ if and only if TX is ample.*

In his amazing proof Mori introduced the use of *rational curves on Fano manifolds.* Since then the method has been much developed to study (uniruled) projective manifolds. The book of J. Kollár, see [16], is a wonderful reference for many fundamental results, I will frequently refer to it (in particular in section V.3, *Characterization of* $\mathbb{P}^n$, one can find the proof of the above theorem).
I will focus on $\mathbb{P}^n$ but I will also present some characterizations of other simple projective varieties as the hyperquadrics and the projective bundles (scrolls) over a smooth variety.

The paper is an extended version of the talk I gave at the Fano conference in Torino, october 2002; I would like to thank and to congratulate the organizers of this beautiful conference.

1 Generalizations of Mori's theorem (over $\mathbb{C}$).

The Mori's theorem and its proof were object of a great interest; soon after Mori himself published another paper ([18]) in which the techniques were developed to prove the famous Cone Theorem. It was the beginning of the so called Minimal Model Program or Mori Theory, whose aim is to provide a (first) classification of projective varieties. More simply one can apply all the new machinery to obtain other possible characterizations of $\mathbb{P}^n$, this is what the results of this section are about. Let me remark that although they were proved only last year, they were conjectured since some years and partial results were provided by many authors.

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In this section X is a complex projective manifold of dimension n. The first theorem, due to J. Wiśniewski and myself, is a natural generalization of the Mori's theorem and it was first conjectured by Campana and Peternell (see [7]).

Theorem 1.1 ([6]) *($k = \mathbb{C}$) $X = \mathbb{P}^n$ if and only if there exists a subsheaf $E \subset TX$ which is locally free and ample; moreover E can be either $T\mathbb{P}^n$ or $\oplus^r \mathcal{O}_{\mathbb{P}}(1)$.*

The second theorem is due to Cho-Miyaoka-Shepherd Barron and to Kebekus; it is a natural generalization of the theorem of Mori since the assumption on the ampleness of TX implies immediately that $-K_X{\cdot}C \geq (n+1)$ for every rational curve C and, less immediately via the Cone theorem, that $-K_X{\cdot}C \geq (n+1)$ for all curves $C \subset X$.

Theorem 1.2 ([8], [14]) *($k = \mathbb{C}$) $X = \mathbb{P}^n$ if and only if $-K_X{\cdot}C \geq (n+1)$ for any curve $C \subset X$.*

It is possible to write a relative formulation of the above two results. A scroll is the projectivization of a vector bundle E over a smooth projective manifold Y, $\varphi : X = \mathbb{P}(E) \to Y$.

Theorem 1.3 ([1]) *($k = \mathbb{C}$). Let $\varphi : X \to Y$ be a surjective map to a normal variety Y with $m := dimY < n$ and let $E \subset T_{X/Y}$ be a locally free subsheaf of the relative tangent bundle.*

1) If E is ample then $X = \mathbb{P}^n$.

2) If E is φ-ample and φ is equidimensional and smooth then $\varphi : X \to Y$ is a scroll.

Theorem 1.4 *($k = \mathbb{C}$). Let $\varphi : X \to Y$ be a surjective map to a normal variety Y with $m := dimY < n$. Assume that φ is equidimensional and smooth and that for every curve $C \subset F$ in a fiber $-K_X{\cdot}C = n - m + 1$. Then $\varphi : X \to Y$ is a scroll.*

Sketch of proof of 1.1 and of 1.2. The proofs of the two theorems are different in many respects. Nevertheless I will try to give a unified presentation of both, skipping many technical details, for which I will refer to the original papers.
First let us recall some notations, they are coherent with the ones used in [16], to which we refer for full details.
Let $Hom(\mathbb{P}^1, X)$ be the scheme representing the functor of generically injective morphisms from $\mathbb{P}^1$ into X; let $RatCurves^n(X)$ be the normalization of its quotient by $Aut(\mathbb{P}^1)$ and let $u : Hom(\mathbb{P}^1, X) \to RatCurves^n(X)$ be the quotient map (see I.1.10, resp. II.2.15, resp. II.2.15 of [16])
We will denote usually by $V \subset Hom(\mathbb{P}^1, X)$ an irreducible closed subvariety which is closed under $Aut(\mathbb{P}^1)$, that is $V = u^{-1}(u(V))$, and by $H \subset$

$RatCurv^n(X)$ its image in $RatCurv^n(X)$. We will say that V or H is unsplit if H is proper (see IV.2.1 of [16]).
Let x be a general point of X; we denote by $V_x = V \cap Hom(\mathbb{P}^1, X; 0 \mapsto x)$, respectively H_x, the family of morphisms in V which map 0 into x, respectively its image in $RatCurv^n(X)$. $\tilde{H}_x$ denotes the normalization of H_x.
Finally by $Locus(V)$ (or $Locus(H)$) we denote the set of points of X which are in the image of a morphism $f \in V$.

First step. We start applying two famous criteria to claim that X is *uniruled*. For the theorem 1.2 we use the Mori's result on the existence of rational curves (see for instance II.5.8 and IV.1.13 in [16]). For the theorem 1.1 we apply the Miyaoka's theorem on varieties with tangent bundle not generically seminegative (see for instance IV.1.16 in [16]).
So we can choose a closed irreducible component $V \subset Hom(\mathbb{P}^1, X)$ which covers X, that is $Locus(V) = X$; let H, V_x, H_x and $\tilde{H}_x$ be as above. By the Mori's bend and break trick we can take V to be *generically unsplit* or *minimal* (see IV.2.1 and IV.2.4 in [16]).
Actually our assumptions imply that for any $f \in V$ we have that $degf^*(E) = rankE := r$ or $r+1$ in 1.1, while $degf^*(-K_X) = n+1$ in 1.2. Thus no curve of V can be split into a sum of rational curves, that is the family V is *unsplit* (see lemma 0.1 in [6]).

Second step. The family V defines a relation of *rational chain connectedness with respect to* V (rcV) in the following way: $x_1,\ x_2 \in X$ are in the rcV relation, $x_1 (rcV)\, x_2$, if there exists a chain of rational curves in V which joins x_1 and x_2. This is an equivalence relation and its equivalence classes can be parameterized generically by an algebraic set by the following result of Campana and Kollár-Miyaoka-Mori.

Theorem 1.5 ([16], IV.4.16) . *There exists an open subset $X^0 \subset X$ and a proper surjective morphisms with connected fibers $\varphi^0 : X^0 \to Z^0$ onto a normal variety and the fibers of φ^0 are equivalence classes of the rcV relation.*

Third step. Assume that Z_0 is just a point, that is X is *rationally connected with respect to the family* V (i.e. there only one equivalence class).
Let x be a general point of X. As done by S. Mori in [17] we consider a *derivative map*

$$\Phi_x : \tilde{H}_x \to P(T_x X) = P((f^*TX)_0)$$

$$\Phi_x([f]) = [(df)_0(\partial/\partial t)]$$

which associates a curve through x which is smooth at x with its tangent direction at x. This is a rational map which was recently studied also by J.M. Hwang and N. Mok.
S. Kebekus proved the following important result which clarifies the structure of Φ_x.

Theorem 1.6 ([13], theorem 3.3 and theorem 3.4) . *Let $x \in X$ be a general point, $H \subset RatCurv^n(X)$ an irreducible family (not necessarily proper) such that $Locus(H) = X$ and H_x is proper.*

1) The subfamily H_x^{Sing} of singular curves through x has dimension at most one. The subfamily $H_x^{sing,x}$ of curves which are singular at x is at most finite and if not empty the associated curves are immersed.

2) The map Φ_x is a regular map and it is finite.

Moreover under our assumption Φ_x is also surjective.
In fact this is clear in the case of theorem 1.2 since $dim H_x = -K_X \cdot l - 2 = n-1$, where l is any curve in H_x (see chapter II of [16]).
For the theorem 1.1 this is more tricky. First we have to divide the proof in two subcases, depending whether $deg f^*(E) = r$ or $r+1$. In the third section of [6] we consider the first case and we show that it is enough to prove the theorem for $r = 1$. Theorem 1.1 was actually proved for $r = 1$ by J. Wahl (see [19]) and thus we conclude. However it would be interesting to produce a direct proof of the surjectivity of Φ_x in this case. In the fourth section of [6] we prove that the second case actually implies that $E = TX$; thus $dim H_x = -K_X \cdot l - 2 = n - 1$ as above and Φ_x is surjective.

A delicate argument now shows that if Φ_x is surjective then it is birational, and thus a biholomorphism. For this argument we refer the reader to the section 3 of [14] (see also [8]). The idea of the argument is due to Miyaoka who suggested to prove that if Φ_x is not generically injective then there exists a curve in V very singular at x, contradicting the theorem 1.6.
The fact that if Φ_x is a biholomorphism then X is $\mathbb{P}^n$ is straightforward and was first noticed by Mori in [17] (see also 3.2 in [14] or V. 3.7.8 in [16]).

Fourth step. Assume that $dim(Z_0) > 0$.
We first note that this case does not occur in the theorem 1.2. In fact in this case $dim H_x = -K_X \cdot l - 2 = n - 1$ and therefore $Locus(H_x) = X$, that is X is rationally connected with respect to the family V.
For the theorem 1.1 in the last section of [6] we show that φ^0 can be extended in codimension 2. Subsequently φ^0, restricted over a general projective curve in the image, is a $\mathbb{P}^k$-bundle and E is a subsheaf of the relative tangent bundle.
But this is a contradiction since $\mathbb{P}^k$-bundle do not have subsheaves of the relative tangent bundle which are ample, by an argument of Campana-Peternell (see [7]).

Theorem 1.3 has been proved in [1] and Theorem 1.4 can be proved similarly as follows.
Proof of 1.4. The proof is by induction on $m := dim Y$. If $m = 0$ then this is the theorem 1.2. Let us assume that the theorem is true for $m-1$. Then the locus of points in Y over which the fiber is not $\mathbb{P}^r$, call it D, is discrete. In fact, take a general very ample divisor A of Y and let $X' = \varphi^{-1}(A) \to A$; this map is by induction a $\mathbb{P}^r$-bundle and therefore $D \subset Y \setminus A$. Thus we can apply Lemma (3.3) of [3] to have an ample line bundle on X whose restriction to a

general fiber is $\mathcal{O}(1)$. The theorem follows then from the Fuijta characterization of scrolls, see [9] lemma (2.12).

I conclude this section with two remarks.
First I think that Theorems 1.3 and 1.4 should be true even without assuming that the morphism φ is smooth.
Second, the above theorems are proved under the assumption that $k = \mathbb{C}$. On the other hand Mori's theorem is true over any algebraically closed base field k and I think the same should hold for the above ones.

2 Adjunction theory.

In this section we will discuss some characterization theorems for $\mathbb{P}^n$ and other simple varieties in positive characteristic, i.e for projective varieties X defined over any algebraically closed base field k.
In order to prove results in this generality we have to restrict ourself to the case of *polarized varieties*, i.e. projective varieties on which an ample vector bundle (polarization) is given. Sometime the study of polarized varieties is called *adjunction theory*.

A main technical tool is the following lemma proved by Y. Kachi and J. Kollár. It is an interesting characterization by itself, which roughly says that if lines through a point cover X, plus a numerical condition, then $X \cong \mathbb{P}^n$ (if $k = \mathbb{C}$ the numerical condition is not needed).

Lemma 2.1 (Lemma 8 in [12]) . *Let X be a normal projective variety defined over an algebraically closed field k and x be a smooth point. Let L be an ample line bundle on X and $H \subset RatCurv^n(X)$ an irreducible component. Assume that*

1) $Locus(H_x) = X$

2) $(C.L) = 1$ for $C \in H_x$

3) $(-K_X.L^{n-1}) > (n-1)(L^n)$.

Then $X \cong \mathbb{P}^n$ and $H = \mathcal{O}_{\mathbb{P}^n}(1)$.

Let us remark that the above lemma is true without the condition 3 in the case $k = \mathbb{C}$, as proved in Lemma 1 of [2]. Moreover S. Kebekus in [13] proved that if $k = \mathbb{C}$ the lemma is true without the condition 3 and replacing condition 2 with the following:

2') $(C.L) = 2$ for $C \in H_x$ and H_x is proper

(of course in this case H will be $\mathcal{O}_{\mathbb{P}^n}(2)$).

The above lemma, together with the Cone Theorem of Mori, implies the following result which was proved in the case of a line bundle (i.e. $r = 1$) by Y. Kachi and J. Kollár and then extended by myself to the case of vector bundle.

Theorem 2.2 ([12] and [1]) . *Let E be an ample vector bundle of rank r on a smooth projective variety X.*

1) *If $r \geq n$, then $K_X + detE$ is nef unless $X = \mathbb{P}^n$ and $E = \oplus^n \mathcal{O}_{\mathbb{P}}(1)$.*

2) *If $r = n-1$ then either $K_X + detE$ is nef or the pair (X, E) is one of the following:*

   i) *$X = \mathbb{P}^n$ and $E = \mathcal{O}_{\mathbb{P}}(2) \oplus (\oplus^{n-2}\mathcal{O}_{\mathbb{P}}(1))$,*

   ii) *$X = \mathbb{P}^n$ and $E = \oplus^{n-1}\mathcal{O}_{\mathbb{P}}(1)$,*

   iii) *X is a quadric $\mathbb{Q}^n \subset \mathbb{P}^{(n+1)}$ and $E = \oplus^{n-1}\mathcal{O}_{\mathbb{P}}(1)$,*

   iv) *X is the projectivization of a rank n vector bundle over a smooth curve A, $\varphi : X \to A$, and $E = \varphi^*(\varphi_*(E(-1))) \otimes \mathcal{O}_{\mathbb{P}}(1)$.*

Sketch of proof of part 1). Let $r \geq n$ and assume that $(K_X + detE)$ is not nef. By the cone theorem, therefore, we have an extremal ray $R = \mathbb{R}^+[C]$ generated by a rational curve C such that $-K_X{\cdot}C \leq (n+1)$ and $(K_X + detE){\cdot}C < 0$. Since $detE{\cdot}C \geq r$ for every rational curve C, all this implies that

$$-K_X{\cdot}C = (n+1) \qquad detE{\cdot}C = r = n.$$

Let $H \subset RatCurv^n(X)$ be an irreducible component which contains C and let $V = u^{-1}(H)$. Since $detE{\cdot}C = r$ the family is proper and since $dimH_x = -K_X{\cdot}C - 2 = n-1$ then $Locus(H_x) = X$.
Thus X is rationally chain connected with respect to the unsplit family V. A very nice property of these varieties is given by the following lemma which is an immediate consequence of the Proposition IV.3.13.3 of [16].

Lemma 2.3 (see [16], IV.3.13.3) *Let X be a projective variety which is rationally chain connected with respect to an unsplit family V. Then the Picard number of X, $\rho(X) =: dimN_1(X)_{\mathbb{R}}$, is 1.*

Therefore let $L := (-K_X + detE)$. $L{\cdot}C = 1$ and thus in particular, by the above lemma, it is ample. We conclude by applying lemma 2.1.

The proof of the part 2) of the theorem is not so straightforward but it involves similar techniques.

I want to remark that the case $k = \mathbb{C}$ was proved since a while by Fujita, Ionescu and Sommese if $r = 1$ and by Peternell, Ye-Zhang for vector bundle.

3 Castelnuovo's type contraction theorems.

In this section we will denote by R an *extremal ray* in the Mori cone of the projective variety X; we define the *length of* R, $l(R)$, by

$$l(R) := \min\ \{-K_X.C : [C] \in R\}.$$

R is said to be *non nef* if $R \cdot D < 0$ for some effective irreducible divisor $D \subset X$; one can notes that R is non nef if and only if $\mathrm{Locus}(R) \neq X$, actually $\mathrm{Locus}(R) \subset D$ (see III.1.6 in [16]).

The next theorem was proved by Occhetta and myself and it is a characterization of the blow-up of a smooth point. For $n = 2$ this is exactly a reformulation of Castelnuovo's contraction theorem.

Theorem 3.1 ([4]) *Let X be a smooth projective variety defined over an algebraically closed field k of dimension n. X has a non nef extremal ray R with $l(R) \geq (n-1)$ if and only if there is a morphism $\varphi : X \to X'$ into a smooth projective variety X' which is the blow-up of X' at a point.*

Sketch of proof. The only if part is obvious. Let then C be a rational curve in the ray R such that $-K_X.C = l(R)$ and let $H \subset RatCurv^n(X)$ be an irreducible component which contains C.
By assumption $\mathrm{Locus}(H) \subset D$, where D is an irreducible divisor, and $-K_X.C = l(R) \geq (n-1)$. On the other hand, as said above, $dim H_x = -K_X \cdot C - 2 \geq n-1$. Thus $-K_X \cdot C = n-1$ and $\mathrm{Locus}(H) = \mathrm{Locus}\ (H_x) = D$. Let $V = u^{-1}(H)$, where u is the map defined in section 1; H is proper, therefore V is said to be unsplit, and $dim V = 2n-1$, the last follows from [16, Theorem II.1.3 and Proposition IV.2.5].
Let $f' : \mathbb{P}^1 \to C \subset D$ be a curve of the family which intersects the smooth locus of D; since every element of $\mathrm{Hom}_{[f']}(\mathbb{P}^1, D)$ is also an element of $\mathrm{Hom}_{[f']}(\mathbb{P}^1, X)$ and $\mathrm{Locus}(V) = D$ we can take an irreducible component of $\mathrm{Hom}_{[f']}(\mathbb{P}^1, D)$, call it W, which is contained in V; this implies that $2n-1 = \dim V \geq \dim W$. On the other hand, since D is a locally complete intersection, being a divisor in a smooth variety, we can apply [16, Theorem II.1.3] which gives $\dim W \geq -K_D.\Gamma + n - 1$; combining the two inequalities we get $-K_D.\Gamma \leq n$.
Recalling that $D.R < 0$, by the adjunction formula $K_D = (K_X + D)_{|D}$, we have that $D.C = -1$.
Using the lemmas 2.1 and 2.3, with some technical work which is due to the fact that we have to pass through the normalization of D, we conclude that $(D, D_D) = (\mathbb{P}^{n-1}, \mathcal{O}(-1))$.
The conclusion is now straightforward.

One can formulate the theorem 1.2 in the following way: $\mathbb{P}^n$ is the only projective variety with an extremal ray of length bigger or equal to $n+1$. Similarly the above theorem says that if a variety has a non nef ray of length $\geq (n-1)$ then it is the blow up of a smooth point in another variety.

In the special case of characteristic zero we extend the classification of non nef extremal rays with high length and we proved the following theorem.

Theorem 3.2 ([4]) *Let X be a smooth n-dimensional projective variety over $\mathbb{C}$ and $R = \mathbb{R}_+[C]$ a non nef extremal ray. Let $\varphi : X \to X'$ be the elementary contraction associated to R and $D = Exc(\varphi)$ its exceptional locus. If $l(R) = n-2$ then one of the following cases occur:*

1) $\varphi(D)$ is a point and $(D,-D_D)\simeq(\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}}(2))$.
2) $\varphi(D)$ is a point and $(D,-D_D)\simeq(\mathbb{Q}^{n-1},\mathcal{O}_{\mathbb{Q}}(1))$, where $\mathbb{Q}^{n-1}$ is a possibly singular quadric.
3) X' is smooth and φ is the blow-up along a smooth curve $\varphi(D)\subset X'$.

The proof of the theorem is similar to the one given above for the first part; in the second part we use a very important result proved in [5] which is about the base point freeness of line bundles. The proof of this last result works only in the case $k=\mathbb{C}$, however it should be nice to have a proof of the above theorem which works for arbitrary characteristic.
For $n=3$ the above theorem 3.2 was proved by S. Mori in [18] and extended by J. Kollár to the case of arbitrary characteristic in [15].

4 Some open questions.

We have described some characterizations of $\mathbb{P}^n$ and we have already found some open questions (basically concerning the extension of some results in positive characteristic).
There are other characterizations of $\mathbb{P}^n$, or rather good properties of projective varieties, which are still conjectures.
For instance we recall the famous **Conjecture of Fujita** (stated in the eighties) which says the following. Let X be a projective manifold of dimension n and L be an ample line bundle. $(X,L)=(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}}(1))$ if and only if $K_X+(n+1)L$ is not very ample.
It is easy to prove the conjecture if L is assumed to be very ample.

Another interesting, and may be less challenged, conjecture is the following one due to Hwang in [11]. The projective space $\mathbb{P}^n$ is, among the Fano manifold with $Pic=\mathbb{Z}$, the one with largest possible automorphism group (with respect to the dimension).

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