Lifting of extremal rays of the Mori cone from an ample section

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Abstract

Let $\mathcal{E}$ be an ample vector bundle of rank r on a complex projective manifold X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s = 0)$ is a smooth submanifold of the expected dimension $\dim X - r := n - r$. We study the problem of lifting birational contractions of Z to the ambient variety, proving an extension property for blow-ups and we apply our results to classify X as above such that Z is a $\mathbb{P}$-bundle on a surface with nonnegative Kodaira dimension.

1 Introduction

Let X be a smooth complex projective variety of dimension n and $\mathcal{E}$ an ample vector bundle of rank r on X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s = 0)$ is a smooth submanifold of the expected dimension $\dim Z = \dim X - r = n - r$.

A classical and natural problem is to ascend the geometric properties of Z to get informations on the geometry of X; for an account of the results in case $r = 1$, i.e. when Z is an ample divisor see [5, Chapter 5]. In [1] we considered the problem from the point of view of Mori theory, posing the following question. Assume that Z is not minimal, i.e. Z has at least one extremal ray in the negative part of the Mori cone and an associated extremal contraction, and that $\dim Z \geq 2$ (if $\dim Z = 1$ then $Z \simeq \mathbb{P}^1$ and this case is treated in [12], where first the problem of special sections of an ample vector bundle was studied); does this ray (or the associated contraction) determine a ray (or a contraction) in X and if so, does this new ray determine the structure of X?

We prove that there is always an extremal face F_Z of Z that determines an extremal face F_X of X (we say that the face is liftable to X); this is the context of Theorem 3.2 and Corollary 3.3, which slightly generalize results in section 3 of [1].

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Note that, a priori, the liftability of a face does not imply the liftability of the contraction; namely the contraction associated to F_X on X restricted to Z is not necessarily the contraction associated to F_Z (see remark 3.4). If this is the case we say that the face is c-liftable. If F_Z corresponds to a fiber type contraction on Z, then F_Z is c-liftable and the contraction of the face F_X in X is again a fiber type contraction [1, 3.12 and 3.13].

In this paper we study the liftability problems for birational contractions: a general result is given in 3.7, then we prove an extension property for blow-ups, namely:

Theorem 1.1 *Let $X, \mathcal{E}$ and Z be as above Let R_Z be an extremal ray on Z, whose associated contraction $\varphi : Z \to Z'$ is the blow-up of a smooth subvariety C of codimension $m \geq 3$.*
Then, if R_Z is liftable to R_X then it is c-liftable; moreover, if $\phi : X \to X'$ is the contraction associated to R_X then X' is smooth, Z' is isomorphic to a subvariety of X', and ϕ is the blow-up of C.

A straightforward corollary of this theorem is the following

Corollary 1.2 *Let $X, \mathcal{E}$ and Z be as above and assume also that $\varphi : Z \to Z'$ is the blow-up of a smooth minimal variety ($K_{Z'}$ is nef) along a smooth subvariety of codimension $m \geq 3$. Then X is the blow-up of a smooth variety X' along a smooth subvariety of codimension $m + r$.*

Another nice application is the following:

Proposition 1.3 *Let $X, \mathcal{E}$ and Z be as above and assume also that Z is the blow-up of a projective space $\mathbb{P}^{n-r}$ along a linear space Y of codimension $m \geq 3$; then X is a projective bundle on $\mathbb{P}^{m-1}$ and $\mathcal{E}_f \simeq \oplus^r \mathcal{O}(1)$ for every fiber of the bundle projection.*

We finally apply our results to prove the following

Theorem 1.4 *Let $X, \mathcal{E}$ and Z be as above and assume also that Z is a $\mathbb{P}^d$-bundle on a surface of nonnegative Kodaira dimension. Then X is a $\mathbb{P}^{r+d}$-bundle on S.*

In the case $r = 1$ the theorem, together with previous existing results, completes the proof of a long lasting general conjecture by A. Sommese for $n = 4$, see [5, Conjecture 5.5.1]; the same theorem was proved in a different way (only for the case $r = 1$ and $n = 4$) recently in [14].

2 Notations and preliminaries

We use the standard notation from algebraic geometry, in particular it is compatible with that of [9] and of [10]. This paper is a sort of sequel of [1] to which we refer constantly for the general set-up and definitions.

The following lemma is probably well known but we will provide anyway a proof for the interested reader, since we did not find a good reference for it.

Lemma 2.1 *Let $\varphi : X \to X'$ be the blow-up of a smooth variety along a smooth subvariety $Y \subset X'$ with exceptional divisor $E(\varphi)$ and $\mathcal{E}$ a rank r vector bundle on X. Assume that $\mathcal{E}_F \simeq \oplus^r \mathcal{O}_{\mathbb{P}}(1)$ for every fiber of φ. Then there exists a rank r vector bundle $\mathcal{E}'$ on X' such that*

$$\mathcal{E} \otimes E(\varphi) = \varphi^* \mathcal{E}'.$$

Proof. The vector bundle $\tilde{\mathcal{E}} := \mathcal{E} \otimes E(\varphi)$ is trivial along any fiber of φ. We have to prove that $f_*(\tilde{\mathcal{E}}) =: \mathcal{E}'$ is a locally free sheaf of rank r. This is a local question at any point $y \in Y$, and we can apply the Theorem on Formal Functions, see [8, Theorem III.11.1], which says that

$$f_*(\tilde{\mathcal{E}})_y = \varprojlim H^0(F_n, \tilde{\mathcal{E}}_n)$$

(with $F_n = X \times_Y Spec(\mathcal{O}_y/m_y^n)$).

$\tilde{\mathcal{E}}$ is trivial on $F = \varphi^{-1}(z)$, that is there are r linearly independent non zero sections of $\tilde{\mathcal{E}}_{|F} = F \times \mathbb{C}^r$; the same proof of Castelnuovo criterion for blow-up, as for instance in [4, Proposition 2.4] or [8, Theorem V.5.7], gives that the sections actually extend to r non zero and linearly independent sections of the $\varprojlim H^0(F_n, \tilde{\mathcal{E}}_n)$ (i.e. they extend in a formal neighborhood of F). This gives that $f_*(\tilde{\mathcal{E}})_y$ is locally free of rank r. □

In the set-up of the previous lemma, it is useful to find conditions which ensure the ampleness of $\mathcal{E}'$:

Lemma 2.2 *In the same assumptions of the previous lemma if $\mathcal{E}$ is ample and Y is a point then also $\mathcal{E}'$ is ample; the same is true if $\mathcal{E}$ is ample and there exists a surjection $N_Y^* \to -L \to 0$ with L nef.*

Proof. The first part has been proved in [13, Lemma 5.1].
To prove the second part we will apply [6, Lemma 5.7]; let $E = \mathbb{P}(N_Y^*)$ be the exceptional divisor of φ and consider the following diagram:

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}') \arrow[d, "p_1"'] & \mathbb{P}(\varphi^*\mathcal{E}') \arrow[l, "\widetilde{\varphi}"'] \arrow[d, "p"] \\ X' & X \arrow[l, "\varphi"'] \end{tikzcd}$$

where $\widetilde{\varphi}$ is the blow-up of $\mathbb{P}(\mathcal{E}')$ along ${p_1}^{-1}(Y)$.
Let $\xi_{\mathcal{E}'}$ be the tautological line bundle of $\mathbb{P}(\mathcal{E}')$; we have

$$\widetilde{\varphi}^* \xi_{\mathcal{E}'} = \xi_{\varphi^*\mathcal{E}'} = \xi_{\mathcal{E}\otimes\mathcal{O}(E)} = \xi_{\mathcal{E}} + p^* E$$

so that $\widetilde{\varphi}^*\xi_{\mathcal{E}'} - p^*E = \xi_{\mathcal{E}}$ is ample on $\mathbb{P}(\varphi^*\mathcal{E}') = \mathbb{P}(\mathcal{E})$.
Moreover $\xi_{\mathcal{E}'}$ is ample on ${p_1}^{-1}(Y)$. In fact, let $Y' \subset E$ be the section corresponding to the surjection $N_Y^* \to -L \to 0$; Y' is mapped isomorphically onto Y by φ and $E_{Y'} = L_{Y'}$, hence we have

$$(\mathcal{E} \otimes L)_{|Y'} = (\mathcal{E} \otimes E)_{|Y'} = \varphi^*\mathcal{E}'_{|Y'} = \mathcal{E}'_{|Y}$$

and $\mathcal{E}'$ is ample on Y.
We can thus apply the quoted lemma to get the ampleness of $\xi_{\mathcal{E}'}$.

3 Lifting of birational contractions

3.1 Assumptions: Let $X, \mathcal{E}$ and Z be as in the introduction, with $dim Z \geq 2$.

This section is dedicated to improve some general results in [1]; we start with a refinement of [1, Theorem 3.4]:

Theorem 3.2 *(Lifting of extremal faces) Assume that Z is not minimal in the sense of Mori theory, i.e. K_Z is not nef. Let F_Z be an extremal face of Z, $D_Z = (K_Z + \tau H_Z)$ a good supporting $\mathbb{Q}$-Cartier divisor of F_Z and H the line bundle on X which restricts to H_Z.*
If H is ample on X the $\mathbb{Q}$-divisor $D = K_X + det\mathcal{E} + \tau H$ is nef but not ample and defines an extremal face F_X of X.
Moreover, if $\dim Z \geq 3$, under the identification $N_1(Z) \simeq N_1(X)$, F_X coincides with F_Z. In this situation we will say that the face F_Z is liftable *to F_X.*

Proof. The first part of the theorem has been proved in [1, Theorem 3.4], so we have to prove only the last assertion.
Since, under the identification $N_1(Z) \simeq N_1(X)$ we have $NE(Z) \subset NE(X)$ and $NE(Z)_{K<0} \subset NE(X)_{K<0}$ to prove our claim it is enough to show that, for every extremal ray R_X in the face F_X, there is a curve in R_X lying on Z.
Let R_X be an extremal ray of F_X and $\varphi_R : X \to T$ the associated extremal contraction; since the contraction of F_X is supported by $K_X + det\mathcal{E} + \tau H$, this divisor is zero on the curves in R_X, yielding $l(R_X) \geq r + \tau$; if the contraction is birational, then, for a non trivial fiber F, $\dim F \geq r + \tau$, hence $\dim F \geq r + 1$ and we are done by [1, Proposition 2.18].
In the same way we get our result if the contraction is of fiber type and has a fiber of dimension $r + 1$; so we are left with the case of an equidimensional fiber type contraction with r-dimensional fibers; note that in this case, the φ_R-ample line bundle H has intersection number one with the extremal rational curve generating the ray by [1, Proposition 2.7], so that, letting $H' = H + \varphi_R^* A$, with A ample on T, the divisor $K_X + (r + 1)H'$ is a good supporting divisor for φ_R, which, by [7, 2.12] is thus a $\mathbb{P}$-bundle contraction.
As in the proof of the first part of [1, Theorem 3.4], we get that Z is a regular section of this bundle, a contradiction with the Lefschetz theorem for ample vector bundles. $\square$

Corollary 3.3 *Let X, $\mathcal{E}$ and Z be as in 3.1; then there exists at least one extremal face F_Z which is liftable to X; moreover, if dim $Z \geq 3$ or, more generally, if $Pic(X) \simeq Pic(Z)$ we can take F_Z to be an extremal ray.*

Proof. Let L be an ample line bundle on X; the restriction of this line bundle to Z, L_Z, is ample on Z, so, if K_Z is not nef there exist a rational number $\sigma > 0$ such that $K_Z + \sigma L_Z$ is nef but not ample and it defines an extremal face G_Z.
This face satisfies the assumptions of theorem (3.2) and so it is liftable to an extremal face G_X; let R_Z be any ray in G_Z and R_X the ray in G_X corresponding to R_Z; let $\psi : X \to Y$ the elementary extremal contraction associated to R_X and A an ample line bundle on Y.
The divisor $K_X + det\mathcal{E} + \sigma L + \psi^* A$ is a good supporting divisor for R_X, and $K_Z + \sigma L_Z + (\Psi^* A)_Z$ is a good supporting divisor for R_Z, so that R_Z is liftable. □

Remark 3.4 Let us note that, a priori, the fact that an extremal face of $\overline{NE(Z)}$ is liftable to an extremal face of $\overline{NE(X)}$ does not imply that the restriction ϕ_Z of the extremal contraction ϕ of F_X coincides with the extremal contraction φ of F_Z; as explained in [1], we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, "\phi"'] & Z \arrow[l, "i"'] \arrow[d, "\varphi"] \arrow[dl, "\phi_Z"'] \\ Y & W \arrow[l, "\pi"] \end{tikzcd} \qquad (3.5)$$

where $\pi : W \to \phi_Z(Z)$ is the normalization morphism.

To complete the lifting process, we introduce the following definition:

Definition 3.6 In the above notation, if $\phi_Z = \varphi$ we will say that the face F_Z is c-liftable.

Clearly a liftable face is c-liftable if and only if $\phi_Z(Z)$ is normal, i.e. if π is the identity. In [1] we proved that if F_Z is a face associated to a fiber type contraction then this is the case; now we will deal with birational contractions.

For the rest of the section, we assume that there exists an extremal ray R_Z which is liftable to an extremal ray R_X, such that the contraction of R_Z, $\varphi : Z \to W$ is supported by $K_Z + \tau H_Z$ and it is birational.

Proposition 3.7 *In the above notation, if φ is birational and $\tau \geq 1$, then ϕ_Z has connected fibers and ϕ is birational, unless $\tau = 1$.*
If $\tau = 1$ then ϕ can be of fiber type, in this case ϕ is a scroll contraction and R_Z is c-liftable.

Proof. If $\tau > 1$ then the same proofs of [1, Propositions 3.13, 3.14] apply. If $\tau = 1$ the contraction ϕ is supported by $K_X + det\mathcal{E} + H$, and so its length $l(\phi)$ is $\geq r+1$.
Supposing that this contraction is of fiber type we have that all its fibers have dimension $\geq r$; if the generic fiber has dimension $\geq r+1$ then it has non trivial intersection with Z, and this is impossible as in the proof of [1, Proposition 3.14], so the generic fiber of ϕ is r-dimensional.
Up to twist H with the pull-back of an ample line bundle, we can assume that $K_X + (r+1)H$ is a good supporting divisor for ϕ which is a scroll contraction (if dim $Z < r+2$ then it is a $\mathbb{P}$-bundle). Note that, in this case $\phi_Z(Z) = \phi(X)$ is normal and so R_Z is c-liftable.

Example 3.8 Let us note that the last case of the above proposition is effective: let $X = \mathbb{P}^2 \times \mathbb{P}^1$ and Z be a $\mathbb{F}_1$ surface in the linear system $\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(1,1)$; the contraction of the (-1) curve of Z lifts to the $\mathbb{P}$-bundle contraction onto $\mathbb{P}^2$.

Proposition 3.9 *If φ and ϕ are both birational, then $E(\varphi) = E(\phi) \cap Z$.*

Proof. If $x \in E(\varphi)$, then there exists a curve $C \subset Z$ which contains x and is contracted by φ; but, on Z, φ and ϕ_Z contract the same curves, therefore $x \in C$ is contained in $E(\phi)$.
On the other hand, if $x \in E(\phi) \cap Z$ we consider the unsplit family V of deformations of an extremal rational curve contracted by ϕ (see [10, IV.2]). By [10, IV.2.6], $\dim \mathrm{Locus}(V, 0 \to x) \geq r + \tau$, hence $\dim(\mathrm{Locus}(V, 0 \to x) \cap Z) \geq 1$, so that x lies in a curve contracted by ϕ, and so by φ. $\square$

Remark 3.10 Actually, the proof of the last proposition shows that the fibers of φ are exactly the intersections of the fibers of ϕ with Z.

4 Blow-ups

Proof of theorem 1.1

Let $D_Z = K_Z + (m-1)H_Z$ be a good supporting divisor of $\varphi : Z' \to Z$, where H_Z is an ample line bundle on Z which restricts to $\mathcal{O}_{\mathbb{P}}(1)$ on every fiber of φ. The contraction ϕ is birational and has connected fibers by 3.7; it is equidimensional, since so is φ, has fibers of dimension $r+m-1$ and it is supported by $D = K_X + det\mathcal{E} + (m-1)H$; we can apply [2, Theorem 5.2] to deduce that $\phi : X \to X'$ is the blow-up of a smooth subvariety of codimension $r+m-1$. Let us point out also that the restriction of $\mathcal{E}$ to every fiber of ϕ splits as $\oplus^r \mathcal{O}_{\mathbb{P}}(1)$; thus by lemma 2.1, we have that $\mathcal{E} \otimes [-E(\phi)] = \phi^*\mathcal{E}'$.

Note that, in view of the discussion in remark 3.4, it is enough to prove that $\phi_Z(Z)$ is smooth and we have to prove it only at points of $\phi_Z(E(\varphi))$; so we consider a local situation: choose a point $z \in \phi_Z(E(\varphi))$, an affine neighbourhood U of z in X' and consider the restrictions of ϕ and φ to the inverse images of

U; to simplify the notation denote again by X, X' and Z the new spaces and by ϕ, ϕ_Z and φ the restricted maps.

In this affine situation and in the notation of the lemma 2.1 we have that $\mathcal{E}'$ is trivial and in particular $\mathcal{E}$ splits as $\oplus^r H$, where $H = -E(\phi)$.
Let H_i, with $i = 1, ..., r$ be general smooth sections of H and let $X_i = \cap_{j=1,\dots,i} H_j$; note that $X_0 = X$ and that $X_r \simeq Z$ (see [3, Lemma 2.6]). We thus have exact sequences

$$0 \to \mathcal{O}_X(D-H) \to \mathcal{O}_X(D) \to \mathcal{O}_{X_1}(D) \to 0$$

$$0 \to \mathcal{O}_{X_1}(D-H) \to \mathcal{O}_{X_1}(D) \to \mathcal{O}_{X_{i+1}}(D_{X_2}) \to 0$$

$$\dots$$

$$0 \to \mathcal{O}_{X_{r-1}}(D-H) \to \mathcal{O}_{X_{r-1}}(D) \to \mathcal{O}_Z(D) \to 0$$

By the Kawamata-Viehveg vanishing theorem we have $H^1(X_i, D_{X_i}) = 0$ for every $i = 1, \dots r$ and thus a chain of surjections

$$H^0(X, D) \twoheadrightarrow H^0(X_1, D_{X_1}) \twoheadrightarrow \cdots \twoheadrightarrow H^0(Z, D_Z).$$

In particular we have that $\phi_Z = \varphi$ and thus $\phi_Z(Z) = \varphi(Z) = Z'$ is smooth and the theorem is proved. $\square$

Proof of proposition 1.3.

The variety Z has two extremal rays: the blow-down contraction to $\mathbb{P}^{n-r}$ and a $\mathbb{P}$-bundle contraction on $\mathbb{P}^{m-1}$; by corollary 3.3 one of these rays lifts to X; if the fiber type ray lifts, then we are done by [1, Corollary 4.2], so we will show that the birational ray cannot lift. Suppose, by contradiction, that this is the case: by theorem 1.1 we have that the ray is c-liftable to a contraction $\phi : X \to X'$ which gives a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\phi"'] & Bl_Y(\mathbb{P}^{n-r}) \arrow[l, "i"'] \arrow[d, "\varphi"] \\
X' & \mathbb{P}^{n-r} \arrow[l, "i"']
\end{tikzcd}$$

We also know that $\mathcal{E}_E = \oplus^r \mathcal{O}(1)$; hence there exists a vector bundle $\mathcal{E}'$ on X' such that $\mathcal{E} = \phi^*\mathcal{E}' \otimes (-E)$; this vector bundle is ample, by lemma 2.2, and $\mathbb{P}^{n-r}$ is the zero locus of a section of it.
This implies, by [12, Theorem A], that X' is a projective space $\mathbb{P}^n$ and $\mathcal{E}'$ decomposes as $\oplus^r \mathcal{O}_{\mathbb{P}^n}(1)$, but this contradicts the ampleness of $\mathcal{E}$. $\square$

5 $\mathbb{P}^d$-bundles on surfaces with $\kappa > 0$ and their Mori cone

Proposition 5.1 *Let $p : Z \to S$ be a $\mathbb{P}^d$-bundle over a smooth surface such that $\kappa(S) \geq 0$; assume that Z has an extremal ray R different from the one associated to the $\mathbb{P}$-bundle contraction. Then the associated contraction φ_R is a blow-down $\varphi_R : Z \to Z'$ of a divisor $E = p^{-1}(C)$, with C exceptional curve on S, such that $E \simeq \mathbb{P}^1 \times \mathbb{P}^d$ and $E_E \simeq \mathcal{O}(0,-1)$.*
Moreover, Z' has a $\mathbb{P}^d$-bundle structure on S', where S' is the surface obtained contracting the exceptional curve C on S, and $\varphi_R(E)$ is a fiber of $p' : Z' \to S'$.

Proof. Suppose that Z has an extremal ray, R, different from the bundle contraction; there exists a rational curve C_0 ($[C_0] \in R$) such that $-K_Z.C_0 > 0$ and $p(C_0)$ is not a point. Let $C = p(C_0)$, let $\nu : \mathbb{P}^1 \to C$ be the normalization of C and consider the fiber product

$$\begin{array}{ccc} Z \times_S \mathbb{P}^1 & \xrightarrow{\bar{\nu}} & Z \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle p} \\ \mathbb{P}^1 & \xrightarrow{\nu} & S \end{array} \tag{5.2}$$

The map $\bar{p} : Z_C := Z \times_S \mathbb{P}^1 \to \mathbb{P}^1$ is a $\mathbb{P}$-bundle on $\mathbb{P}^1$ and so $\rho(Z_C) = 2$; the morphism $\bar{\nu}$ induces a map of spaces of cycles $N_1(Z_C) \to N_1(Z)$ which is an embedding. The Mori cone $NE(Z_C)$ is contained in the intersection $N_1(Z_C) \cap NE(Z)$ and so, since $N_1(Z_C)$ is a plane in $N_1(Z)$ and passes through two different extremal rays of Z, $NE(Z_C)$ is contained in the negative part of $NE(Z)$.
By [11, Corollary 2.8] $-K_{Z_C/\mathbb{P}^1}$ is not ample, so there exist an horizontal curve C_1 on Z_C such that $-K_{Z_C/\mathbb{P}^1}.C_1 \leq 0$; noting that

$$-K_{Z_C/\mathbb{P}^1} = \bar{\nu}^* p^* K_S - \bar{\nu}^* K_Z$$

we get

$$\bar{\nu}^* p^* K_S.C_1 = -K_{Z_C/\mathbb{P}^1}.C_1 + \bar{\nu}^* K_Z.C_1 \leq K_Z.\bar{\nu}(C_1) < 0$$

and therefore $K_S.\varphi(\bar{\nu}(C_1)) = K_S.C < 0$.

Since on S there is only a finite number of curves which have negative intersection with K_S (the (-1)-curves) we deduce that the image of the exceptional locus of R is C, and C is a (-1)-curve.
Moreover, since the fibers of different extremal contractions can meet only in points, we have that all the fibers of φ_R have dimension one; combining these informations we get that φ_R is a divisorial contraction. By [3, Theorem 4.1] φ_R is a smooth blow-down contraction.

The exceptional locus of φ_R, E, is thus $p^{-1}(C)$ and carries two different $\mathbb{P}$-bundle structures, being so forced to be $\mathbb{P}^1 \times \mathbb{P}^d$; the description of E_E is clear observing that the lines in one ruling are extremal curves for the blow-up, while those in the other ruling are contained in fibers of the bundle projection. Let $\mathcal{F}$ be a rank $d+1$ vector bundle on S such that $Z = \mathbb{P}_S(\mathcal{F})$; the restriction of $\mathcal{F}$ to C is, up to twist, $\oplus^{d+1}\mathcal{O}_{\mathbb{P}^1}$, so, denoted by $\sigma : S \to S'$ the contraction of C, by lemma 2.1 there exists a rank $d+1$ vector bundle $\mathcal{F}'$ on S' such that $\mathcal{F} = \sigma^*\mathcal{F}'$. Consider the commutative diagram

$$\begin{tikzcd} Z = \mathbb{P}(\mathcal{F}) \arrow[r, "\bar{\sigma}"] \arrow[d, "p"] & Z' = \mathbb{P}(\mathcal{F}') \arrow[d, "p'"] \\ S \arrow[r, "\sigma"] & S' \end{tikzcd}$$

The map $\bar{\sigma}$ is a good contraction which contracts exactly the curves in R, so it coincides with φ_R. $\square$

6 $\mathbb{P}^d$-bundles on surfaces as ample sections

Proof of theorem 1.4.

By proposition 5.1 the extremal rays of Z are the ray corresponding to the $\mathbb{P}^d$-bundle fibration and, possibly, other rays of birational type, corresponding to blow-downs $\pi : Z \to Z_1$ which contract $\mathbb{P}^d \times \mathbb{P}^1$ to $Y \simeq \mathbb{P}^d$.
By corollary 3.3, we have that at least one extremal ray of Z is liftable to X; if this ray is the fiber type one, then, by [1, Corollary 4.2] X is a $\mathbb{P}^{r+d}$-bundle on S and we are done.

Suppose now that the ray that is liftable is a birational one; by proposition 5.1 Z_1 has a $\mathbb{P}^d$-bundle structure over a smooth surface S_1, obtained from S contracting a (-1)-curve.
By proposition 3.7 if π lifts to a fiber type contraction, ϕ, then ϕ is a scroll contraction $\phi : X \to Z_1$. Take a smooth non rational curve B in S_1 which does not meet the exceptional curves of S_1 and consider in X the smooth subvariety $U = \phi^{-1}(p'^{-1}(B))$, which has a $\mathbb{P}^r$-bundle structure on $p^{-1}(B)$; $Z \cap U$ is the zero locus of a section of $\mathcal{E}_U$, and is a $\mathbb{P}^d$-bundle on a non rational curve; hence, by [1, Corollary 4.4] U is a $\mathbb{P}^{r+d}$-bundle on B, a contradiction.

So π lifts to a birational contraction $\Pi : X \to X_1$ that, by (1.1) is a smooth blow-up of $Y_1 \subset X_1$, such that the restriction of $E(\Pi)$ to Z is $E(\pi)$; thus by lemma 2.1 there exists a vector bundle $\mathcal{E}_1$ on X_1 such that

$$\mathcal{E} \otimes \mathcal{O}_X(E) = \Pi^*\mathcal{E}_1$$

and moreover, by lemma 2.2, $\mathcal{E}_1$ is ample.

Summing up, we have replaced the starting triple $(X, \mathcal{E}, Z)$ with a new triple $(X_1, \mathcal{E}_1, Z_1)$ satisfying the assumptions of the theorem and such that $\rho(Z_1) = \rho(Z) - 1$. Again, one of the extremal contractions of Z_1 has to lift. Since $\rho(Z)$ is finite, at some point of the process we find some triple $(X_k, \mathcal{E}_k, Z_k)$ such that the $\mathbb{P}^d$-bundle contraction of Z_k lifts to a $\mathbb{P}^{r+d}$-bundle contraction of X_k and $\mathcal{E}_{k|F} \simeq \oplus^r \mathcal{O}(1)$ for every fiber of the bundle contraction.

Let $\Pi_k : X_{k-1} \to X_k$ be the last blow-down contraction and let F be the fiber of the $\mathbb{P}^{r+d}$-bundle contraction of X_k which contains Y_k, the center of Π_k (which is $F \cap Z_k \simeq \mathbb{P}^d$). Let l' be a line in F which meets transversally Y_k and let l be its strict transform in X_{k-1}. We have

$$(\mathcal{E}_{k-1})_l \simeq (\Pi_k^* \mathcal{E}_k)_l \otimes \mathcal{O}(-E_k)_l \simeq (\oplus^r \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \simeq \oplus^r \mathcal{O}_{\mathbb{P}^1},$$

contradicting the ampleness of $\mathcal{E}_{k-1}$. □

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