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On contractions of smooth varieties. (English. English summary)

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Contractions of smooth varieties. II. Computations and applications. (English. Italian summary)

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FEATURED REVIEW.

The basic features of the theory of algebraic surfaces were solidly established around the beginning of the 19th century but the first steps of a higher-dimensional theory appeared only in the fundamental paper of S. Mori [Ann. of Math. (2) 116 (1982), no. 1, 133–176; MR 84e:14032]. The basic question of this theory, nowadays called the minimal model program or Mori's program, is the following: Given an algebraic variety X (which we usually assume to be smooth and projective) we would like to find another variety X' which is birational to X such that the global structure of X' is as simple as possible. Moreover, we would like to find a procedure which constructs X' as the final result of a sequence of elementary steps. These steps are analogs of the inverses of blowing-ups of smooth subvarieties.

Mori [op. cit.] gave a complete list of these elementary steps in the case of smooth threefolds. Smoothness is, unfortunately, a strong restriction, since the program cannot be carried out within the category of smooth varieties. One has to allow the so-called terminal singularities introduced by M. Reid [in Journees de Geometrie Algebrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, 273–310, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980; MR 82i:14025]. It was also understood that there are two types of elementary steps in general. A contraction (or a Fano-Mori contraction) is a birational morphism $X \to Y$ such that the canonical class of X is negative on the fibers. In some cases the resulting variety Y is very singular (the precise meaning of this is not important for the papers under review) and then we need another step, called a flip. Flips can be viewed as analogs of Dehn surgeries in the theory of topological 3-manifolds.

An approach to the study of contractions was developed by Y. Kawamata [Ann. of Math. (2) 119 (1984), no. 3, 603–633; MR 86c:14013b] and at least the general features of their behaviour seem to be well understood. The study of flips turned out to be much harder. The existence of flips in the three-dimensional case was established by Mori [J. Amer. Math. Soc. 1 (1988), no. 1, 117–253; MR 89a:14048], as the last piece of the minimal model program in dimension three. Since then most of the progress in the theory has been restricted to dimension three. Many applications of these methods were discovered and these led to the solution of numerous open problems in the theory of surfaces and threefolds. Some of these are reviewed in [J. Kollar and S. Mori, Birational geometry of algebraic varieties, Cambridge Univ. Press, Cambridge, 1998]. There has been very little progress on the theory of minimal models in dimensions four and higher. Recent advances of higher-dimensional algebraic geometry have followed a different path; see Kollar and Mori [op. cit. (Section 7.6)] for a quick survey.

The papers under review reach back to the beginnings of the minimal model program. They achieve in dimension four what Mori [op. cit.; MR 84e:14032] completed for threefolds. In this they constitute the first significant step of the last decade in the four-dimensional minimal model program. The results of these papers build on and partially complete earlier work of Y. Kachi [Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 24 (1997), no. 1, 63–131; MR 98j:14016].

The main result is the theorem (see Part I, p. 256) which states the following. Let $g\colon X \to Z$ be a proper birational morphism of fourfolds. Assume that X is smooth, $-K_X$ is ample on the fibers of g and let F be a two-dimensional fiber such that nearby fibers have dimension at most 1. The F is either the projective plane or a quadric (which may be smooth, a cone or the union of two planes). The conclusion holds scheme theoretically and the theorem also describes the normal bundle of F. If g is not birational, there are many more cases. These are listed in Part I, Proposition 4.11.

The theorem is stated clearly and concisely, but it is not for the uninitiated. First, it is not obvious that its assumptions are reasonable, but I want to assure the reader that the case tackled by the authors is indeed the difficult one. The remaining Fano-Mori contractions of smooth fourfolds are either already understood [Y. Kawamata, Math. Ann. 284 (1989), no. 4, 595–600; MR 91e:14039] or probably not very interesting. Second, as with any long classification result, it is not clear that such a list is indeed useful. An answer to this concern is not yet clear, but I believe that applications of this result will surely come.

To gain some appreciation of the result the reader may prefer to start with Part I, Section 3. There one can find a list of carefully worked out examples of contractions. To me this was the really enjoyable part of the paper. The examples are intricate and many-faceted. It is quite amazing to see that so many do occur. Having seen so many of these, one is equally surprised that there are no more. The hard work of the paper is concentrated in Sections 4 and 5. A very careful study of the contractions using the base point freeness method and the deformation theory of curves leads to a classification.

Part II deals with some of the questions left open in the first part. It contains two main results. Section 1 gives a description of some Fano-Mori contractions of smooth fourfolds by explicit equations. These are actually quite complicated but the new point of view is likely to be useful and certainly instructive. Section 4 studies Fano-Mori contractions where a divisor is contracted to a curve. These contractions were also studied by H. Takagi [Proc. Amer. Math. Soc. 127 (1999), no. 2, 315–321]. This case was not investigated in Part I and, as expected, there are no surprises here. *Janos Kollar* (1-UT)