AN INTRODUCTION TO MORI THEORY: THE CASE OF SURFACES

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Introduction

One of the main results of last decades algebraic geometry was the foundation of Mori theory or Minimal Model Program, for short MMP, and its proof in dimension three. Minimal Model Theory shed a new light on what

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is nowadays called higher dimensional geometry. In mathematics high numbers are really a matter of circumstances and here we mean greater than or equal to 3. The impact of MMP has been felt in almost all areas of algebraic geometry. In particular the philosophy and some of the main new objects like extremal rays, Fano-Mori contractions or spaces and log varieties started to play around and give fruitful answer to different problems.

The aim of Minimal Model Program is to choose, inside of a birational class of varieties, "simple" objects. The first main breakthrough of the theory is the definition of these objects: minimal models and Mori spaces. This is related to numerical properties of the intersection of the canonical class of a variety with effective cycles. After this, old objects, like the Kleiman cone of effective curves and rational curves on varieties, acquire a new significance. New ones, like Fano-Mori contractions, start to play an important role. And the tools developed to tackle these problems allow the study of formerly untouchable varieties.

Riemann surfaces were classified, in the $\mathrm{XIX}^{\mathrm{th}}$ century, according to the curvature of an holomorphic metric. Or, in other words, according to the Kodaira dimension. Surfaces needed a harder amount of work. For the first time birational modifications played an important role. The theory of (-1)-curves studied by the Italian school of Castelnuovo, Enriques and Severi, at the beginning of $\mathrm{XX}^{\mathrm{th}}$ century, allowed to define minimal surfaces. Then the first rough classification of the latter, again by Kodaira dimension, was fulfilled.

Minimal Model Program is now a tool to start investigate this question in dimension 3 or higher. It has its roots in the classical results for surfaces and therefore the surface case is a perfect tutorial case. This was first pointed out by S. Mori who worked out a complete description of extremal rays in the case of a smooth surface (see [Mo, Chapter 2], see also [KM2, pg 21-23, §1.4]). Moreover he also showed how it is possible to associate to each extremal ray a morphism from the surface. When the ray is spanned by a rational curve with self intersection -1, this is a celebrated theorem of Castelnuovo.

The purpose of these notes is to give a summary of MMP or Mori Theory in the case of surface.

I took most of the statements as well as of the proofs from papers and book of M. Reid, S. Mori, J. Kollár and S. Mori, K. Matsuki and O. Debarre. More precisely the first part was largely taken from chapter D of [Re3] while the last part is a revised version of section 5 of [AM].

Part 1. Preliminaries

In this part we collect all definitions which are more or less standard in the algebraic geometry realm in which we live.

1.1. Basic notation

First we fix a good category of objects. Let X be a normal variety over an algebraically closed field k, that is an integral separated scheme which is of finite type over k. We consider the case of dimension 2 (almost all the definitions however work in any dimension n),

We actually assume also that $char(k) = 0$, nevertheless many results hold also in the case of positive characteristic.

We have to introduce some basic objects on X.

Let $Div(X)$ be the group of **Cartier divisors** on X and $Pic(X)$ be the group of **line bundles** on X, i.e. $Div(X)$ modulo linear equivalence. We will freely pass from the divisor to the bundle notation. Let $Z_1(X)$ be the group of 1-cycles on X i.e. the free abelian group generated by reduced irreducible curves.

If X is smooth we denote by K_X the **canonical divisor** of X, that is an element of $Div(X)$ such that $\mathcal{O}_X(K_X) = \Omega^2_X$, where Ω_X is the sheaf of one forms on X. If X has some mild singularities (normal) a canonical sheaf can be defined and it is a line bundle as soon as the singularities are very nice (for instance rational double points).

Then there is a pairing

$$Pic(X) \times Z_1(X) \to \mathbb{Z}$$

defined, for an irreducible reduced curve $C \subset X$, by $(L, C) \to L \cdot C := deg_C(L_{|C})$, and extended by linearity. In the case of surfaces this is the intersection pairing coming from Poincare duality

$$H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}.$$

Two line bundles $L_1, L_2 \in Pic(X)$ are **numerically equivalent**, denoted by $L_1 \equiv L_2$, if $L_1 \cdot C = L_2 \cdot C$ for every curve $C \subset X$. Similarly, two 1-cycles C_1, C_2 are *numerically equivalent*, $C_1 \equiv C_2$ if $L \cdot C_1 = L \cdot C_2$ for every $L \in Pic(X)$.

Define

$$N^1X = (Pic(X)/\equiv) \otimes \mathbb{R} \text{ and } N_1X = (Z_1(X)/\equiv) \otimes \mathbb{R};$$

obviously, by definition, $N^1(X)$ and $N_1(X)$ are dual $\mathbb{R}$-vector spaces and $\equiv$ is the smallest equivalence relation for which this holds.

In particular for any divisor $H \in Pic(X)$ we can view the class of H in $N^1(X)$ as a linear form on $N_1(X)$. We will use the following notation:

$$H_{\geq 0} := \{x \in N_1(X) : H \cdot x \geq 0\} \text{ and similarly for } > 0, \leq 0, < 0$$

and

$$H^{\perp} := \{x \in N_1(X) : H \cdot x = 0\}.$$

The fact that $\rho(X) := dim_{\mathbb{R}} N^1(X)$ is finite is the **Neron-Severi** theorem, [GH, pg 461]. The natural number $\rho(X)$ is called the Picard number of the variety X. (Note that for a variety defined over $\mathbb{C}$ the finite dimensionality of $N_1(X)$ can be read from the fact that $N_1(X)$ is a subspace of $H_2(X, \mathbb{R})$).

Note that since X is a surface then $N^1(X) = N_1(X)$; using M. Reid words (see [Re3]), "Although very simple, this is one of the key ideas of Mori theory, and came as a surprise to anyone who knew the theory of surfaces before 1980: the quadratic intersection form of the curves on a nonsingular surface can for most purpose be replaced by the bilinear pairing between N^1 and N_1, and in this form generalises to singular varieties and to higher dimension."

We notice also that **algebraic equivalence**,see [GH, pg 461], of 1-cycles implies numerical equivalence. This follows from the fact that the degree of a line bundle is invariant in a flat family of curves.

Moreover, if X is a variety over $\mathbb{C}$ then, in terms of Hodge Theory, $N^1(X) = (H^2(X,\mathbb{Z})/(Tors) \cap H^{1,1}(X)) \otimes \mathbb{R}$.

The following is a standard definition that we will frequently use.

Definition 1.1.1. Let $V \subset \mathbb{R}^n$ be a closed convex cone. A subcone $W \subset V$ is called *extremal* if $u, v \in V, u + v \in W \Rightarrow u, v \in W$. A one dimensional subcone is called a ray.

1.2. Morphisms associated to line bundles; ample line bundles

Let L be a line bundle (or a divisor) on X. L is said the be **base point free**, or **free** for short, if for every point $x \in X$ the restriction map $H^0(X, L) \to H^0(x, L_x) = \mathbb{C}$ is surjective. In particular the rational map defined by the global section of a line bundle, which we will denote by φ_L, is a regular map if the line bundle is base point free. L is said to be **semiample** if lL is free for $l >> 0$.

Theorem 1.2.1 (Stein factorization and Zariski's main theorem)**.** *Let L be a semiample line bundle and let $\varphi' := \varphi_{lL} : X \to W'$ be the associated morphism. Then φ' can be factorized into $g \circ \varphi$ such that $\varphi : X \to W$ is a projective morphism with connected fibers and $g : W \to W'$ is a finite morphism. Moreover*
i) $LC = 0$ for a curve $C \subset X$ iff C is in a fiber of φ
ii) W is a normal variety or equivalently $\varphi'_*(\mathcal{O}_X) = \mathcal{O}_W$ (Zariski's main theorem)
iii) φ coincides with φ_{lL} for $l >> 0$.

Definition 1.2.2. A **contraction** is a surjective morphism $f : X \to W$, with connected fibers, between normal varieties. If L is a semiample line bundle we will in general consider its **associated contraction** as the part with connected fibers in the Stein factorization of the map φ_{lL}.

L is said to be **very ample** if it base point free and if the map by φ_L is an embedding. L is **ample** if mL is very ample for some positive $m \in \mathbb{N}$.

We introduce now some criteria for the ampleness of a line bundle as well as some very useful vanishing theorems. Later we will introduce new ones which are more related to our subject.

Theorem 1.2.3 (Serre, Criterium and Vanishing)**.** *L is ample iff any of the conditions below holds:*

- *for any coherent sheaf F on X*
$$H^i(X, F \otimes \mathcal{O}_X(nL)) = 0$$
for $i > 0$ and $n >> 0$.
- *for any coherent sheaf F on X, $F \otimes \mathcal{O}_X(nL)) = 0$ is generated by its global sections for $n >> 0$.*
- *for any line bundle H on X, $H \otimes \mathcal{O}_X(nL)) = 0$ is very ample for $n >> 0$.*

Theorem 1.2.4 (Kodaira, Criterium and Vanishing)**.** *Let L be a line bundle on X.*

- *L is ample iff there exists on L an hermitean metric with curvature form Θ such that $(\frac{i}{2\pi})\Theta$ is positive.*
- *If L is ample then*
$$H^i(X, K_X + L)) = 0$$
for $i > 0$.

Theorem 1.2.5 (Nakai- Moishezon, Criterium)**.** *Let L be a line bundle on X. L is ample iff $L^{dim Z} \cdot Z > 0$ for every irreducible subvariety $Z \subset X$ (for surfaces this is $L^2 > 0$ and $LC > 0$ for every curve $C \subset X$).*

1.3. Riemann - Roch and Hodge Index theorems

Let $D \in Div(X)$ and $H \in Pic(X)$.

Theorem 1.3.1 (Riemann - Roch)**.**
$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X)$$
and
$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X)) = \frac{1}{12}(c_1^2 + c_2),$$
where $e(X)$ is the topological Euler number, i.e. the alternating sum of Betti numbers.

An easy consequence of the RR theorem is the following

Corollary 1.3.2. *$D^2 > 0$ implies either $h^0(nD) \neq 0$ or $h^0(-nD) \neq 0$ for $n >> 0$. The sign of HD, for H ample, distinguish the two cases.*

Proof. By R.R.
$$h^0(nD) + h^2(nD) \geq \chi(\mathcal{O}_X(nD)) \sim n^2 D^2/2,$$
so either $h^0(nD)$ or $h^2(nD)$ goes to infinity with n. Now by Serre duality
$$h^2(nD) = h^0(K_X - nD).$$
The same argument work with D replaced by $-D$, therefore we have also that either $h^0(-nD)$ or $h^0(K_X + nD)$ goes to infinity with n.

On the other hand $h^0(K_X - nD)$ and $h^0(K_X + nD)$ cannot both go to infinity with n: indeed if $h^0(K_X - nD) \neq 0$ multiplying by a non zero section $s \in H^0(K_X - nD)$ gives an inclusion $H^0(K_X + nD) \to H^0(2K_X)$ so that $h^0(K_X + nD) \leq h^0(2K_X)$.

Therefore either $h^0(nD)$ or $h^0(-nD)$ grows quadratically with n.

Corollary 1.3.3 (the Index theorem)**.** *If H is ample on X then $HD = 0$ implies $D^2 \leq 0$. Moreover if $D^2 = 0$ then $D \equiv 0$.*

In other words the intersection pairing on N^1X has signature $(+1, -(\rho - 1)$. The positive part correspond to a (real) multiple of H.

Another way of stating this is that if D_1, D_2 are divisors and $(\lambda D_1 + \nu D_2)^2 > 0$ for some $\lambda, \nu \in \mathbb{R}$ then the determinant

$$\begin{vmatrix} D_1^2 & D_1D_2 \\ D_1D_2 & D_2^2 \end{vmatrix} \leq 0.$$

Proof. If $D^2 > 0$ then either nD or $-nD$ is equivalent to a non zero effective divisor for $n >> 0$. Since H is ample, either of these conditions implies $HD \neq 0$. This proves the first statement.

Assume that $D^2 = 0$ and by contradiction that $D \not\equiv 0$; thus there exists a curve $A \subset X$ such that $DA \neq 0$. Replace A by $B = a - \lambda H$ so that $HB = 0$. Now if $DA \neq 0$ also $DB \neq 0$ and some linear combination of D and B has $(D + \alpha B)^2 > 0$. This contradicts the first part.

Part 2. Mori theory for surfaces

2.1. Mori-Kleiman cone

We denote by $NE(X) \subset N_1(X)$ the cone of **effective 1-cycles**, that is

$$NE(X) = \{C \in N_1(X) : C = \sum r_i C_i \text{where } r_i \in \mathbb{R}, r_i \geq 0\},$$

where C_i are irreducible curves. Let $\overline{NE(X)}$ be the closure of $NE(X)$ in the real topology of $N_1(X)$. This is called the *Kleiman–Mori cone*.

We also use the following notation:

$$\overline{NE(X)}_{H \geq 0} := \overline{NE(X)} \cap H_{\geq 0} \text{ and similarly for } > 0, \leq 0, < 0.$$

One effect of taking the closure is the following trivial observation, which has many important use in applications: if $H \in N^1(X)$ is positive on $\overline{NE(X)} \setminus 0$ then the section $(H \cdot z = 1) \cap \overline{NE(X)}$ is compact. Indeed, the projectivised of the closed cone $\overline{NE(X)}$ is a closed subset of $\mathbb{P}^{\rho-1} = P(N_1(X))$, and therefore compact, and the section $(H \cdot z = 1)$ projects homeomorphically to it. The same holds for any face or closed sub-cone of $\overline{NE(X)}$.

An element $H \in N^1(X)$ is called **numerically eventually free** or **numerically effective**, for short **nef**, if $H \cdot C \geq 0$ for every curve $C \subset X$ (in other words if $H \geq 0$ on $\overline{NE(X)}$).

The relation between nef and ample divisors is the content of the Kleiman criterium that is a corner stone of Mori theory. Let $D, H \in Pic(X)$.

Corollary 2.1.1 (weak form of the Kleiman criterium)**.** *If D is nef on X and H is ample then*
i) $D^2 \geq 0$
ii) $D + \epsilon H$ *is ample for any* $\epsilon \in \mathbb{Q}$, $\epsilon > 0$.

Proof. The quadratic polynomial $p(t) = (D+tH)^2$ is a continuous increasing function for positive $t \in \mathbb{Q}$ and $p(t) > 0$ for sufficiently large t. We **claim** that if $p(t) > 0$ for positive $t \in Q$ then also $p(\frac{t}{2}) > 0$. This will actually implies i). In fact since $(D+tH)^2 > 0$ (by assumption) and $H(D+tH) > 0$ (H is ample) by the Corollary 1.3.2 we have that $n(D+tH)$ is effective for suitable $n >> 0$. Since D is nef $(D + \frac{t}{2}H)^2 = D(D+tH) + \frac{t}{2}^2 H^2 > 0$.

To prove ii) we use the Nakai-Moishezon criterium. In fact $(D+\epsilon H)\Gamma > 0$ for every curve Γ and $(D+\epsilon H)^2 > 0$ (the last from the above discussion).

Theorem 2.1.2 (Kleiman criterium)**.** *For $D \in Pic(X)$, view the class of D in $N^1(X)$ as a linear form on $N_1(X)$. Then*

$$D \text{ is ample} \Longleftrightarrow DC > 0 \text{ for all } C \in \overline{NE(X)} \setminus \{0\}.$$

In other words the theorem says that the cone of ample divisors is the interior of the nef cone in $N^1(X)$, that is the cone spanned by all nef divisors.

Note that it is not true that $DC > 0$ for every curve $C \subset X$ implies that D is ample, see for instance Example 3 in the next section. The condition in the theorem is stronger.

This is only a weak form of Kleiman's criterion, since X is a priori assumed to be projective. The full strength of Kleiman's criterion gives a necessary and sufficient condition for ampleness in terms of the geometry of $\overline{NE(X)}$.

The statement as well as the following proof of the theorem work in all dimension. Moreover it holds even if D is a $\mathbb{Q}$ divisor.

Proof. The implication $\Longleftarrow$ follows from part ii) of the above corollary. Choose a norm $|| \;\; ||$ on $N_1(X)$. The set $K = \{z \in \overline{NE(X)} : ||z|| = 1\}$ is compact. The functional $z \to Dz$ is therefore bounded from below on K by a positive rational number a; if H is ample the functional $z \to Hz$ is bounded from above on K by a positive rational number b. Therefore $(D - \frac{a}{b}H)$ is non negative on K and thus on $\overline{NE(X)}$. In particular $(D - \frac{a}{b}H)$ is nef and $D = (D - \frac{a}{b}H) + \frac{a}{b}H$ is ample.

For the other implication $\Longrightarrow$ note that the intersection with D is a positive linear functional on $NE(X)$, thus it is non negative on $\overline{NE(X)}$. Assume that $Dz = 0$ for some $z \in \overline{NE(X)} \setminus \{0\}$. Let H be a line bundle such that $Hz < 0$. Then $H + kD$ is ample for $k >> 0$, thus

$$0 \leq (H+kD)z = Hz < 0$$

a contradiction.

A great achievement of Mori theory is a description of $\overline{NE(X)}$. We start giving some properties of it.

Proposition 2.1.3. *$\overline{NE(X)}$ does not contain a line and if H is an ample divisor then the set*

$$\{z \in \overline{NE(X)} : Hz \leq k\}$$

is compact, hence it contains only finitely many classes of irreducible curves.

Proof. Let H be ample. If the cone $\overline{NE(X)}$ contains a line then this line should pass through the origin. But the the intersection with H is a linear functional on the line positive outside the origin, a contradiction. For the second part let $D_1, ..., D_\rho$ be divisor on X such that $[D_1], ..., [D_\rho]$ is a basis for $N^1(X)$. There exists an integer m such that $mH \pm D_i$ is ample for each i. Thus for any $z \in \overline{NE(X)}$ we have $|D_i z| \leq mHz$. If $Hz \leq k$ this bounds the coordinates of z and defines a closed bounded set. It contains at most finitely many classes of irreducible curves because the set of these classes is by construction discrete in $N_1(X)$.

Lemma 2.1.4. *The set $Q := \{D \in N_1(X) : D^2 > 0\}$ has two (open) connected components,*

$$Q^+ := \{D \in Q : HD > 0\} \text{ and } Q^- := \{D \in Q : HD < 0\},$$

where H is an ample line bundle. Furthermore $Q^+ \subset \overline{NE(X)}$. In particular if $[D] \in \overline{NE(X)}$ and if $D^2 > 0$ then $[D]$ is in the interior of $\overline{NE(X)}$.

Proof. By the Index theorem the intersection on $N_1(X)$ has exactly one positive eigenvalue. In a suitable basis it can be written as $x_1^2 - \Sigma_{i\geq 2} x_i^2$; moreover we can choose that base such that $[H] = ((HH)^{1/2}, 0, ..., 0)$. The two connected components are therefore defined by

$$Q^+ := \{x_1 > (\Sigma_{i\geq 2} x_i^2)^{\frac{1}{2}}\} \text{ and } Q^- := \{x_1 < -(\Sigma_{i\geq 2} x_i^2)^{\frac{1}{2}}\}$$

By 1.3.2 for every $D \in Q$ either D or $-D$ is effective. Since effective curves have positive intersection with H they are in Q^+.

Lemma 2.1.5. *Let $C \subset X$ be an irreducible curve in a surface X. If $C^2 \leq 0$ then $[C]$ is in the boundary of $\overline{NE(X)}$. If $C^2 < 0$ then $[C]$ is extremal in $\overline{NE(X)}$.*

Proof. If $D \subset X$ is an irreducible curve such that $D \cdot C < 0$ then $D = C$. If $C^2 = 0$ then $D \to D \cdot C$ is a linear functional which is non negative on $\overline{NE(X)}$ and zero on C.

In general $\overline{NE(X)}$ is spanned by $\mathbb{R}_{\geq 0}[C]$ and $\overline{NE(X)}_{C\geq 0}$, because the class of every irreducible curve $D \neq C$ is in $\overline{NE(X)}_{C\geq 0}$. If $C^2 < 0$ then $[C] \notin \overline{NE(X)}_{C\geq 0}$ and thus $[C]$ generates an extremal ray.

Lemma 2.1.6. *Let $[D] \in \overline{NE(X)}$ be an extremal ray. Then either $D^2 \leq 0$ or $\rho(X) = 1$. If $D^2 < 0$ then the extremal ray is spanned by the class of an irreducible curve.*

Proof. The first part follows at once by the Lemma 2.1.4. For the second one let D_n be a sequence of effective 1-cycles converging to $[D]$. Note that for $n >> 0$ $D_n^2 < 0$; Thus there is an irreducible component E_n of $supp(D_n)$ such that $E_n^2 < 0$. By the previous Lemma 2.1.5 we have that $E_n \in [D]$.

2.2. Examples

1) $\mathbb{P}^2$, the projective plane: in this case $N_1(X) = \mathbb{R}$ and there is nothing to say.
2) $\mathbb{P}_1^1 \times \mathbb{P}_2^1$, the smooth quadric. $N_1(X) = \mathbb{R}^2$ and $\overline{NE(X)} = NE(X) = \mathbb{R}^+ l + \mathbb{R}^+ l'$ where l is a line in $\mathbb{P}_1^1$ and l' a line in $\mathbb{P}_2^1$. Each ray corresponds to the projection on the other factor.
3) Let X be a minimal surface over a smooth curve of genus g. We will use many facts about ruled surfaces whose proof can be find in [Ha]. The vector space $N_1(X)$ has dimension 2. It is generated by the class of a fiber f and the class of a certain section C_0. The ray generated by the class $[f]$ is extremal since it is the relative subcone associated with the projection $\pi : X \to C$.

Set $e := -C_0^2$. This is an invariant of X which can take any value $\geq -g$. When $e \geq 0$ any irreducible curve on X is numerically equivalent to C_0 or to $aC_0 + bf$ with $a \geq 0$ and $b \geq ae$. In particular,

$$\overline{NE(X)} = NE(X) = \mathbb{R}^+[C_0] + \mathbb{R}^+[f].$$

The extremal ray $\mathbb{R}^+[f]$ is associated with the projection, the other extremal ray, $\mathbb{R}^+[C_0]$, may or may not be contracted. When $g = 0$ the contraction exists: it is the morphism associated with the base point free linear system $|C_0 + ef|$ and, when $e > 0$, it is birational and contracts only the curve C_0 to a point on a surface (note that for $e = 0$ we have $X = \mathbb{P}_1^1 \times \mathbb{P}_2^1$).

When $e < 0$ (thus $g > 0$) and the characteristic is zero any irreducible curve on X is numerically equivalent to C_0 or to $aC_0 + bf$ with $a \geq 0$ and $2b \geq ae$. Moreover any divisor $aC_0 + bf$, with $a > 0$ and $2b > ae$ is ample, hence some multiple is the class of a curve. This implies

$$\overline{NE(X)} = \mathbb{R}^+[2C_0 + ef] + \mathbb{R}^+[f].$$

When $g = 1$ the only possible value is $e = -1$ and there is a curve numerically equivalent to $2C_0 + ef$: the cone $NE(X)$ is closed.

When $g \geq 2$ and the base field is $\mathbb{C}$ there exists a rank 2-vector bundle $\mathcal{E}$ of degree 0 on C all of whose symmetric powers are stable. The normalization of $\mathcal{E}$ has even positive degree $-e$. For the associated ruled surface $\mathbb{P}(\mathcal{E})$ no multiple of the class $[2C_0 + ef]$ is effective. In particular it is not ample although it has positive intersection with every curve on X (this was an example first presented by Mumford, for further details see [Ha2], ch. I section 10). Note that $[C_0 + e/2f] = 1/2[2C_0 + ef]$ is the class of the tautological bundle $\mathcal{O}_X(1)$. Moreover $(2C_0 + ef)^2 = 0$ (otherwise it would be ample for the Nakai Moishezon criterium) and the Mori-Kleiman cone

$NE(X) = \mathbb{R}^+[2C_0 + ef] + \mathbb{R}^+[f]$ is not closed (otherwise $[2C_0 + ef]$ would have been ample for Kleiman criterium).
4) Let A be an abelian surface and H an ample line bundle on it. Note that A does not contain rational curves and that K_A is trivial, in particular the self intersection of any curve of A is non negative. It follows from 1.3.2 that $\overline{NE(A)}$ is given by the condition $D^2 \geq 0$ and $DH \geq 0$. Therefore if $\rho(A) \geq 3$ (e.g. if $A = E \times E$, E an elliptic curve) then $\overline{NE(A)} = \overline{Q^+}$ is a circular cone. Every point in the boundary of $\overline{NE(A)}$ is extremal, most of these point have irrational coordinates and thus they do not correspond to any curve on A.
5) Let X be a del Pezzo surface, i.e. a surface with $-K_X$ ample. One can prove that X is either isomorphic to X_k, the blow up of $\mathbb{P}^2$ in $k = 0, 1, ..., 8$ "general" points or it is $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover X_3 is a smooth cubic surface in $\mathbb{P}^3$. A part from $\mathbb{P}^2$ we will see that one can find a finite number of rational curve $C_1, ..., C_r$ such that $C_i^2 \leq 0$ and $NE(X) = \mathbb{R}^+[C_1] + ... + \mathbb{R}^+[C_r]$, in particular $\overline{NE(X)} = NE(X)$. (For further details see the end of the next section).
6) (Nagata's example-1960) Let $X \to \mathbb{P}^2$ be the blow up of the nine base points of a general pencil of cubics in $\mathbb{P}^2$. Let $\pi : X \to \mathbb{P}^1$ be the morphism given by the pencil of cubics and let B the finite subset of X where π is not smooth. The exceptional divisors $E_0, ..., E_8$ are section of π. Smooth fibers of π are elliptic curves hence become abelian groups by choosing their intersection with E_0 as zero. Translations by elements of E_i then generate a subgroup of $Aut(X \setminus B)$ isomorphic to $\mathbb{Z}_8$. It is easy to see that any automorphism of $(X \setminus B)$ can actually be extended to an automorphism σ of X. Moreover for any such σ, the curve E_σ is a (-1) curve. So X has infinitely many (-1) curves all of which span an extremal ray of $\overline{NE(X)}$ (2.1.5). Finally note that $|-K_X|$ is the elliptic pencil, so $-K_X$ is nef but not ample.
7) (Zariski's example-1962) Let $g : X \to \mathbb{P}^2$ be the blow up of the twelve points $p_1, ..., p_{12}$ on a smooth cubic plane curve D. Let $C \subset X$ be the birational transform of D. $C^2 = -3$, therefore by the Grauert criterium (1962), C can be contracted via an analytic morphism $f : X \to Y$ to an analytic surface Y. However Y cannot be projective if the 12 points are in general position. To see this, suppose M is any line bundle on Y. Then $f^*(M)$ is linearly equivalent to $g^*\mathcal{O}(b) + \Sigma a_i E_i$ where E_i are the exceptional divisors above p_i. But $f^*(M)_{|C}$ is linearly equivalent to zero, therefore we would have a linear equivalence $\mathcal{O}(b) + \Sigma a_i p_i$ on D which is clearly impossible for general choice of p_i.

However if the p_i are the points of intersection of a quartic curve Q with D, then the linear system $|M|$ spanned by Q and by the quartics of the form $C + (line)$ is birationally transformed to a free linear system $g_*^{-1}|M|$ and it realises $f : X \to Y$ as a projective morphism.

The example shows that there can be no numerical criterion for contractibility in the projective category (but for extremal rays in the negative part of the cone, as we will see in a next section).

2.3. The rationality lemma and the cone theorem

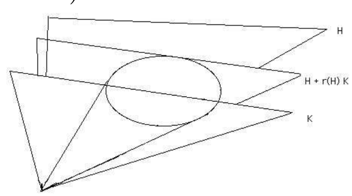
A combinative use of Riemann Roch theorem and of (Kodaira) vanishing theorem gives the following result. It is easy to be proved in the case of surface and it becomes much more intricate in higher dimension; in particular then it needs a more sophisticated vanishing theorem (Kawamata-Vieheweg).

Theorem 2.3.1 (Rationality theorem)**.** *Let X be a smooth surface for which K_X is not nef. Let H be an ample line bundle on X. Define the nef threshold (or nef value) of L by*

$$t_0 = t(H) = \sup\{t \in \mathbb{R} : tK_X + H \text{ is nef}\}.$$

Then
i) the nef threshold is a rational number and
ii) its denominator is ≤ 3.

Remark 2.3.2. The condition K_X is not nef means that the half space $\{K_X z < 0\}$ of $N_1(X)$ meets $\overline{NE(X)}$. To interpretate the theorem one should think (see the picture)



at the family of linear systems $tK_X + H$ starting from the initial position $t = 0$ outside $\overline{NE(X)}$ (since H is ample) and rotating to its asymptotic position for $t >> 0$. t_0 is the value at which it first hits $\overline{NE(X)}$.

Proof. The proof is in three steps.
Step 1. If $n(H + t_1 K_X) = D_1$ is effective for some $n > 0$, with $t_1 \in \mathbb{Q}$ and $t_1 > t_0$ (that is D_1 is not nef), then t_0 is determined by

$$t_0 = \min_{\Gamma \subset D_1} \{\frac{H\Gamma}{-K_X\Gamma}\}$$

where the minimum runs over the irreducible components of Γ of D_1 such that $K_X\Gamma < 0$. In fact for $0 < t < t_1$ the divisor $H + tK_X$ is a positive combination of H and D_1 (namely $H + tK_X = H + \frac{t}{t_1}(t_1 K_X) =$

$(1-\frac{t}{t_1})H + \frac{t}{t_1}(\frac{1}{n}D_1))$ so that it fails to be nef if and only if $(H+tK_X)\Gamma < 0$ for a component Γ of D_1.

Step2. Now if $t_0 \notin \mathbb{Q}$ then, for $n, m \in \mathbb{Z}$ with $n < mt_0 < n+1$, it follows that $mH + nK_X$ is ample but $mH + (n+1)K_X$ is not nef. For $m > 0$ set $mt_0 = n + \alpha$, with $0 \le \alpha < 1$, and write $D_0 = H + t_0K_X \in N^1(X)$. Then Kodaira vanishing theorem and RR give

$$H^0(mH + (n+1)K_X) \ge \chi(\mathcal{O}_X) + 1/2(mH + (n+1)K_X)(mH + nK_X) =$$

$$\chi(\mathcal{O}_X) + 1/2(m^2D_0^2 + m(1-2\alpha)D_0K_X - \alpha(1-\alpha)K_X^2).$$

Hence if $D_0^2 > 0$ then $H^0 \neq 0$ for large m.

If $D_0^2 = 0$ and $D_0 \not\equiv 0$ then necessarily $D_0K_X < 0$ (this because $D_0(H + t_0K_X) = 0$ and $D_0H > 0$) and therefore $H^0 \neq 0$ if m is large and $(1-2\alpha)$ is bounded away from 0.

If $D_0^2 = D_0K_X = 0$ then also $D_0H = 0$, so that $D_0 \equiv 0$ and this implies that $K_X \equiv -(1/t_0)H$ (in particular $-K_X$ is ample).

All this, via step 1, proves (i).

Step 3. We have to prove part ii), namely that if $t_0 = \frac{p}{q}$ with $p \in \mathbb{N}$ then $q \le 3$. This is postponed after the proof of the base point freeness theorem (2.5.4).

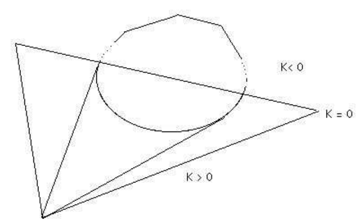
The first main theorem of Mori theory is the following description of the negative part, with respect to K_X, of the Kleiman-Mori cone.

Theorem 2.3.3 (Cone theorem)**.** *Let X be a non singular surface.Then*

$$\overline{NE(X)} = \overline{NE(X)}_{K_X \ge 0} + \sum R_i$$

where R_i are extremal rays of $\overline{NE(X)}$ contained in $\overline{NE(X)}_{K_X<0}$. Moreover for any ample divisor H and $\epsilon > 0$ there are only finitely many extremal rays R_i such that $(K_X + \epsilon H)R_i \le 0$

In simple words the theorem says the following. Consider the linear form on $N_1(X)$ defined by K_X; the part of the Kleiman-Mori cone $\overline{NE(X)}$ which sits in the negative semi-space defined by K_X (if not empty) is locally polyhedral and it is spanned by a countable number of *extremal rays*, R_i. Moreover moving an ϵ away from the hyperplane $K_X = 0$ (in the negative direction) the number of extremal rays becomes finite.



There are essentially two ways of proving this theorem; the original one, which is due to Mori, is very geometric and valid in any characteristic. It is presented in the paper [Mo] and in many other places, for example in [KM2] and [De]. It is based on the study of deformations of rational curve on an algebraic variety, it makes use of the theory of Hilbert schemes.

It was noticed by M. Reid and Y. Kawamata that the rationality theorem (and the Base point free theorem) implies immediately the Mori's cone theorem, in the more general case of varieties with LT singularities. We will give a proof following the second approach; the argument is pure convex body theory.

Preliminaries to the prove of the cone theorem. Let us fix a basis of N^1X of the form $K_X, H_1, ..., H_{\rho-1}$, where the H_i are ample and ρ is the Picard number. As in the rationality lemma, for any nef element $L \in N^1X$, set

$$t_0(L) = \max \{t | L + tK_X \text{ is nef } \}.$$

If L is nef and the corresponding face $F_L = F^{\perp} \cup \overline{NE(X)}$ is contained in $\overline{NE(X)}_{K_X<0}$ (for instance if L is ample) then the rationality lemma gives $6t_0(L) \in \mathbb{Z}$.

Lemma 2.3.4. *Let $L \in Pic(X)$ be a nef divisor which supports a face F_L contained in $\overline{NE(X)}_{K_X<0}$. Consider $\nu L + H_i$ for all i and $\nu >> 0$.*
1) $t_0(\nu L + H_i)$ is an increasing function of ν, is bounded above, and attains its bound.
2) Let ν_0 be any point after $t_0(\nu L + H_i)$ has attained its upper bound and suppose $\nu > \nu_0$. Set

$$L'_i = 6(\nu L + H_i + t_0(\nu L + H_i)K_X)$$

(multiplying by 6 is simply to ensure that $L'_i \in Pic(X)$ by the rationality lemma). Then L'_i supports a face $F_{L'_i} \subset F_L$.
3) If $dim F_L \geq 2$ then there exists i and $\nu >> 0$ such that

$$L' = 6(\nu L + H_i + t_0(\nu L + H_i)K_X)$$

supports a strictly smaller face $F_{L'_i} \subsetneq F_L$.
4) In particular F_L *contains an extremal ray* R *of* $\overline{NE(X)}$.
5) If $F_L = R$ *is an extremal ray and* $z \in R$ *is a nonzero element then* $\frac{6H_iz}{K_Xz} \in \mathbb{Z}$.
6) The extremal ray in $\overline{NE(X)}_{K_X<0}$ *are discrete.*

Proof. 1) is almost obvious. $t_0(\nu L + H_i)$ is an increasing function of ν by construction. It is bounded above since for any point $z \in F_L \setminus \{0\}$ we have $Lz = 0$ and $(H_i + tK_X)z < 0$ for $t >> 0$. It attains its bound because t_0 varies in the discrete set $\frac{1}{6}\mathbb{Z}$.

2) Suppose that $t_0 = t_0(\nu L + H_i)$ does not change with $\nu \geq \nu_0$. Then for $\nu > \nu_0$ any $z \in F_{L'_i}$ satisfies

$$0 = (\nu L + H_i + t_0K_X)z = (\nu - \nu_0)Lz + (\nu_0 L + H_i + t_0K_X)z,$$

and therefore $Lz = 0$ (a priori both terms in the last part are ≥ 0).

3) Consider $\nu >> 0$ and

$$L'_i = 6(\nu L + H_i + t_0(\nu L + H_i)K_X)$$

for each i. Since $\nu >> 0$ these are small wiggles of L in $\rho - 1$ linearly independent directions. Each $F_{L'_i} \subset F_L$ is a face of F_L. The intersection of all the $F_{L'_i}$ is contained in the set defined by

$$(\nu L + H_i + t_0(\nu L + H_i)K_X)z = 0$$

which are $\rho - 1$ linearly independent conditions on z. Therefore at least one of the $F_{L'_i}$ is strictly smaller than F_L.

4) follows obviously from 3).

5) follows from 2) and the rationality theorem. Indeed since $F_L = R$ is a ray, 2) implies that $F_{L'_i} = F_L = R$. That is R is orthogonal to $H_i + t_0K_X$ and $6t_0 \in \mathbb{Z}$.

6) follows from 5) since in $(K_Xz < 0)$ every ray contains a unique element z with $K_Xz = -1$ and $H_iz \in (\frac{1}{6})\mathbb{Z}$.

Proof. (of the Cone theorem). Let $B = \overline{NE(X)}_{K_X \geq 0} + \sum R_i$. Note that $B \subset \overline{NE(X)}$ is a closed convex cone. Indeed by 6) he R_i can have accumulation points only in $\overline{NE(X)}_{K_X \geq 0}$.

Suppose $B \subsetneq \overline{NE(X)}$: Then there exists an element $M \in N^1X$ which is nef and supports a non zero face F_M of $\overline{NE(X)}$ disjoint from B, necessarily contained in $\overline{NE(X)}_{K_X<0}$. By the usual compactness argument, for sufficiently small $\epsilon > 0$, $M - \epsilon K_X$ is ample, and $M + \epsilon K_X$ is not nef but positive on B. These are all open condition on M, and any open neighbourhood of M in N^1X contains a rational element M, so that, by passing to this and taking a multiple, I can assume that $M \in PicX$. But now the rationality lemma and 4) imply that there is a ray R of $\overline{NE(X)}$ with $MR < 0$, so that $R \not\subset B$. This contradicts the choice of B and proves the cone theorem.

Proposition 2.3.5. *For every extremal ray R such that $R \cdot K_X < 0$ there exists a nef divisor H_R such that $H_R \cdot z = 0$ if and only if $z \in R$. (This fact is true in any dimension).*

Proof. This follows immediately from the cone theorem and the rationality theorem. Roughly one take an element $M \in N^1(X)$ which satisfies the assumption of the proposition. Then in a small neighbourhood of M one can find an ample line bundle $D \in Pic(X)$. Then H is given by $D + r(D)K_X$.

Proposition 2.3.6. *If X has an extremal ray R contained in $\overline{NE(X)}_{K_X<0}$ with $R^2 > 0$ then $X = \mathbb{P}^2$.*

Proof. By the Lemma 2.1.6 we have that $\rho(X) = 1$. Let H be an ample generator of $N^1(X)_{\mathbb{Z}}$ modulo torsion and write $-K_X \equiv rH$ for $r > 0$. In particular X is a del Pezzo surface : Thus it follows immediately from Kodaira vanishing theorem and Serre duality the following.

Lemma 2.3.7. *Let X be a del Pezzo surface, then $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$ and $\chi(X, \mathcal{O}_X) = 1$.*

By Poincaré duality $H^2 = 1$ and $b_2(X) = 1$. Thus $c_2(X) = e(X) = 3$. By Noether formula $c_1(X)^2 = 9$ and therefore $r = 3$. By RR

$$h^0(X, \mathcal{O}_X(H)) = \chi(X, \mathcal{O}_X(H)) = \frac{1+3}{2}H^2 = 2.$$

Since $H^2 = 1$, $|H|$ has no base points and so it defines a degree 1 non constant morphism to $\mathbb{P}^2$. Thus it is an isomorphism.

Exercise 2.3.8. *Prove that for a del Pezzo surface the cone $\overline{NE(X)}$ is as described in the example 5 above.*

Remark 2.3.9. By lemma 2.1.6 and proposition 2.3.6 we have that if R is an extremal ray contained in $\overline{NE(X)}_{K_X<0}$ with $R^2 \neq 0$ then R is spanned by the class of an irreducible curve.

However the complete version of Mori cone theorem says that every extremal ray R contained in $\overline{NE(X)}_{K_X<0}$ is spanned by the class of an irreducible and *rational* curve (even if $R^2 = 0$). This is actually one of the most remarkable achievement of Mori's theory.

The proof of this fact follows immediately in the case of surface by the precise description of the contraction associated to any R given in 2.5.7.

Corollary 2.3.10. *Let X be a smooth surface with TX ample. Then $X = \mathbb{P}^2$.*

Proof. Since TX is ample then also $-K_X = \Lambda^2 TX$ is ample. Thus we have extremal rays R contained in $\overline{NE(X)}_{K_X<0}$. If $R^2 > 0$ then we conclude by the above proposition 2.3.6. Let us assume by contradiction that $R^2 \leq 0$ and let $C \in R$ be an effective irreducible curve. By adjunction formula

$$2g(C) - 2 = K_X C + CC < 0.$$

Thus C is rational and $-K_X C \leq 2$; this is a contradiction with the next lemma.

Lemma 2.3.11. *Let $C \subset X$ be a rational curve on a smooth surface X and let $i : \mathbb{P}^1 \to C \subset X$ its resolution. If $i^*TX_{|C}$ is ample then $-K_X C \geq 3$.*

Proof. $i^*TX_{|C}$ is ample and therefore $i^*TX_{|C} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ for $a_1 \geq a_2 > 0$. Since the natural homomorphism $T_{\mathbb{P}^1} = \mathcal{O}(2) \to i^*TX_{|C}$ is non zero then $a_1 \geq 2$. Thus $-K_X C = a_1 + a_2 \geq 3$

2.4. Castelnuovo contraction theorem

In this section we state the famous Castelnuovo theorem. The reader should look at the very interesting proof of the theorem for instance in [Ha], V.5.7.

An irreducible curve $C \subset X$ in a smooth surface X is called a (-1) curve if $C^2 = K_X C = -1$. Note that by adjunction formula a (-1) curve is a smooth rational curve.

Theorem 2.4.1 (Castelnuovo contraction theorem)**.** *Let X be a smooth projective surface, $C \subset X$ a (-1) curve. Then there exists a morphism $\varphi : X \to X'$ onto a smooth projective surface X' such that $\varphi(C) = pt$ and $\varphi : X \setminus C \to X' \setminus pt$ is an isomorphism.*

φ is called the contraction of C or more often the blow-up of X' at the (smooth) point $x := \varphi(C)$.

The reader can try to prove the following lemma as an exercise or he/she can read the proof for instance on [Ha]:

Lemma 2.4.2. *Let X' be a smooth surface. Let $\varphi : X \to X'$ be the blow up of a point $p \in X'$ and $E \subset X$ be the (-1) curve. Then*
i) $Pic(X) = Pic(X') \oplus \mathbb{Z}$ and $\rho(X) = \rho(X') + 1$.
ii) $f^*(K_{X'}) = K_X - E$

Let $C \subset X'$ be a curve of multiplicity r at p and let $\tilde{C}$ be its strict transform in X. Then
iii) $f^*(C) = \tilde{C} + rE$
iv) $g(\tilde{C}) = g(C) - \frac{1}{2}r(r-1)$

Remark 2.4.3. If D is an effective divisor in X not equal to E then $f_*(D) := D'$ is a well defined divisor in X', by Riemann extension theorem.

If D is ample, resp. nef, the same is for D'. If $D^2 > 0$ then $D'^2 > 0$.

It is not known whether if D is a very ample line bundle then D' is very ample

2.5. Base point freeness, BPF

The aim of BPF is to show that an adjoint linear system, under some conditions, is free from fixed points.

We start with the easy case of the curve. Let C be a compact Riemann surface of genus g, let K_C be the canonical bundle of C and let H be a line bundle.

Theorem 2.5.1. *If $degH \geq 2g$ then H has no base point.*

Proof. Let $L := H - K_C$ and let $x \in C$ be a point on C. Note that by assumption $degL \geq 2$ and thus

$$H^1(C, K_C + L - x) = H^0(C, x - L) = 0, \tag{2.5.1}$$

the first equality coming from Serre duality.

Then we consider the exact sequence

$$0 \to \mathcal{O}_C(K_C + L - x) \to \mathcal{O}_C(K_C + L) \to \mathcal{O}_x(K_C + L) \to 0,$$

which comes by tensoring the structure sequence of x on C,

$$0 \to \mathcal{I}_x \to \mathcal{O}_C \to \mathcal{O}_x \to 0,$$

by the line bundle $K_C + L$.

The sequence gives rise to a long exact sequence in cohomology whose first terms are, keep in mind equation (2.5.1),

$$0 \to H^0(C, K_C + L - x) \to H^0(C, K_C + L) \to H^0(x, K_C + L) \to 0.$$

In particular we have the surjective map

$$H^0(C, H) \xrightarrow{\alpha} H^0(x, H) \to 0.$$

Furthermore x is a closed point and therefore

$$H^0(x, H) = \mathbb{C} \neq 0. \tag{2.5.2}$$

The surjectivity of α translates into the existence of a section of $\mathcal{O}(H)$ which is not vanishing at x. That is the pull back via α of 1. □

Remark 2.5.2. The above can be used to prove the following:

Theorem 2.5.3. *Let L be an ample line bundle on a curve C (i.e. $degL > 0$), then $K_C + 2L$ is spanned by global sections.*

What we have done can be summarised in the following slogan, which is somehow the manifesto of the base point free technique.

Construct section of an adjoint line bundle proving a vanishing statement, (2.5.1), and a non vanishing on a smaller dimensional variety, (2.5.2).

Let us see a possible generalisation in the case of surface.

Theorem 2.5.4 (Base point freeness)**.** *Let X be a smooth surface and D be a (rational) divisor which is nef but not ample and such that $D - \epsilon K_X$ is ample for some $\epsilon > 0$. Then for all $m \gg 0$ the line bundle mD is generated by global sections.*

Remark 2.5.5. The above theorem is true also in higher dimension and it was proved by Y. Kawamata and VV. Shokurov (see [Ka] and [Sh]) with a method which builds up from the classical methods of the Italians and which was developed in the case of surfaces by Kodaira-Ramanujan-Bombieri.

A very significant step in the understanding and in the spreading out of the technique was given in a beautiful paper of M. Reid (see [Re1]) which we strongly suggest to the reader.

This type of results are fundamental in algebraic geometry and they are constantly under improvement, recently important steps were achieved. among others by Kawamata,Shokurov, Kollár and Ein-Lazarsfeld.

A big drawback is that the method, as it stands, is not effective, i.e. it does not give a good bound for m. This is in fact a problem in higher dimension, in the surface case however we have another method, which woks in slightly different assumptions, and which is in fact very effective. (It is called Reider method. It gives for instance that if X is a smooth surface of general type with $K_S^2 \geq 5$ then $2K_X$ is base point free).

Proof. There are three possible cases.
Case I) If $D^2 > 0$ then there must exists a curve C with $DE = 0$, since otherwise D would be ample by the Nakai Moishezon theorem. Then $C^2 < 0$ by the Index theorem and also $K_X C < 0$. In particular C is a -1-curve, i.e. $K_X C = C^2 = -1$. Note also that the Index theorem implies that any two of these are disjoint. Now we apply Castelnuovo theorem 2.4.1.

From 2.4.2 and the following remark we have that
i) $D' := \varphi_*(D))$ is a nef divisor on X' with $D'^2 > 0$
ii) $\varphi_*(D - \epsilon K_X) = D' - \epsilon K_{X'}$ is an ample divisor.

If D' is not ample then there exists another (-1) curve C' with $D'C' = 0$. We repeat the above process and after a finite number of steps (the process has to be finite by 2.4.2 i)), $g : X \overset{f^1}{\to} X^1 \overset{f^2}{\to} ... \overset{f^j}{\to} X^j$, we reach a smooth surface X^j with an ample divisor D^j. Then $D = g^*(D^j)$ and the theorem is proved.
Case II). $D_0^2 = 0$ and $D \not\equiv 0$. Then since $D - \epsilon K_X$ is ample, $K_X D < 0$. $h^i(mD) = 0$ for $i > 0$ by Kodaira vanishing theorem since $D - \epsilon K_X$ is ample and therefore RR gives $h^0(mD) \sim (-K_X D/2)m$ for $m >> 0$. So in particular $h^0(mD) \to \infty$. Let M and F be respectively the moving and the fixed part of mD. Since M is nef we have

$$0 \leq M^2 \leq M(M+F) = MmD \leq (M+F)mD = (mD)^2 = 0$$

which implies

$$M^2 = MF = F^2 = 0.$$

Thus $|M|$ is base point free, since it has no fixed components and $M^2 = 0$. Let $\varphi_{mM} : X \to W$ be the associated contraction (i.e. the map associated to a high multiple of mM, see 1.2.1), where W is a smooth curve, and let H is a very ample line bundle on W such that $mM = \varphi^*(H)$. Since $MF = 0$ then F has to be contained in a fiber, since $F^2 = 0$ then F has to be actually proportional to a full fiber. Hence $F = \varphi^*(\Sigma a_i P_i)$ for some positive rational number a_i and $P_i \in W$.

Therefore

$$|m'mD| = m'(\varphi^*(H + \Sigma a_i P_i)).$$

Since $(H + \Sigma a_i P_i)$ is ample $|m'mD|$ is base point free for $m' >> 0$.

Case III. $D_0^2 = 0$ and $D \equiv 0$. We prove that $H^0(X, D) \neq 0$, therefore that $D = \mathcal{O}$. For this note that $lD - K_X \equiv -K_X \equiv (D - \epsilon K_X)/\epsilon$ is ample for any l. Thus

$$h^0(X, D) = \chi(\mathcal{O}_X(D) = \frac{1}{2}(D - K_X)D + \chi(\mathcal{O}_X) = h^0(X) = 1.$$

Proof. (**of part ii) of theorem 2.3.1**) We use the notation of the proof of the above theorem 2.5.4 with $D = H + t_0 K_X$ and $\epsilon = t_0$. Again we have three cases.

Case I) $D^2 > 0$: then we have a (-1)-curve C with $DE = 0$. Thus $t_0 = \frac{HC}{-K_X C} = HC$ is an integer.

Case II) $D^2 = 0$ and $D \not\equiv 0$. Let φ_{mD} be the contraction associated to (a high multiple of) D and let F be a general fiber. We have that $F^2 = 0$ and $-K_X F < 0$. By the (arithmetic) genus formula we have that $-K_X F = 2$ and therefore that $t_0 = \frac{HF}{-K_X F}$ has denominator equal to either 1 or 2.

Case III) $D^2 = 0$ and $D \equiv 0$. If $\rho(X) > 1$ let H' be a divisor not proportional to H. Then $D' = H' + t_0(H')K_X$ must be in one of the previous case and thus there is a curve C such that $-K_X C = 1$ or 2. Since $D \equiv 0$ we have $DC = 0$ which implies $t_0 = \frac{HC}{-K_X C}$ has denominator equal to either 1 or 2. If $\rho(X) = 1$ let L be an element in $N^1(X)$ corresponding to 1. Then $-K_X \equiv kL$ and we conclude if $k \leq 3$. Assume then by contradiction that $k > 3$. Then, for $x = 1, 2, 3$,

$$\chi(\mathcal{O}_X(K_X + xL)) = h^0(X, \mathcal{O}_X(K_X + xL)) = 0$$

since $K_X + xL \equiv (x - k)H$. This would imply that $\chi(\mathcal{O}_X(K_X + xL)) = 0$ as a polynomial in x , a contradiction (since $\chi(\mathcal{O}_X(K_X + xL)) > 0$ for $x >> 0$).

We will know apply the above theorem to contract extremal rays in the Mori cone. We start with a definition.

Definition 2.5.6. Let R be an extremal ray of $\overline{NE(X)}$. A projective morphism $\varphi : X \to W$ onto a normal projective variety W such that

i) For any irreducible curve $C \subset X$, $\varphi(C)$ is a point if and only if $C \in R$.
ii) φ has connected fibers
iii) $H = \varphi^*(A)$ for some ample Cartier divisor on W.

is called the **extremal contraction** associated to R.

In general not all the extremal ray can be contracted, i.e. has an associated extremal contraction. An example is provided in section 2.2. example 7.

However for the ray in the negative part of the cone, with respect to K_X, we have the following.

Theorem 2.5.7 (Contraction theorem)**.** *For each extremal ray R in the half space $N_1(X)_{K_X<0}$ there exists the associated extremal contraction $\varphi_R : X \to W$. Moreover φ_R is one of the following types:*

- *(1) Z is a smooth surface and X is obtained from Z by blowing-up a point; $\rho(Z) = \rho(X) - 1$.*
- *(2) Z is a smooth curve and X is a minimal ruled surface over Z; $\rho(X) = 2$.*
- *(3) Z is a point, $\rho(X) = 1$ and $-K_X$ is ample; in fact $X \cong \mathbb{P}^2$.*

Proof. Let H_R be a nef divisor as in 2.3.5. Then by Kleiman's criterion for ampleness there exists a natural number a such that $aH_R - K_X$ is ample.

Now we can apply theorem 2.5.4 and conclude that mH is base point free for $m >> 0$. Let $\varphi : X \to W$ be the associated contraction (i.e. the map associated to a high multiple of mH_R, see 1.2.1). This is the extremal contraction associated to R (for $m >> 0$).

For the second part, in the proof of the base point freeness theorem we have seen that the maps associated to the section of mH_R are of three types.
Type 1). φ_R can be the contraction of a finite number of -1-curves; then it is the contraction of one of them since two different -1 curves are numerically not proportional.
Type 2). φ_R is a conic bundle then actually φ gives the structure of a minimal ruled surface. In order to prove this we have to show that there are no reducible or non reduced fiber of φ In fact if, by contradiction, F is such a fiber then $F = \sum a_i C_i = [C]$ with $[C] \in R$. But since R is extremal this implies that $C_i \in R$ for every i. Thus $C_i^2 = 0$, since a general fiber of φ is a smooth irreducible and reduced curve in the ray, and $C_i \cdot K_X < 0$. By the adjunction formula this implies that $C_i \simeq \mathbb{P}^1$ and $C_i \cdot K_X = -2$. Thus

$$-2 = (C \cdot K_X) = \sum a_i (C_i \cdot K_X) = -2 \sum a_i,$$

which gives a contradiction. Furthermore using Tsen's Theorem, one can prove that X is the projectivization of a rank two vector bundle on $\mathbb{P}^1$, see [Re3, C.4.2].
Type 3). φ_R is the contraction of a del Pezzo surface then this surface must have Picard number one and therefore it is $\mathbb{P}^2$ by (the proof of) Proposition 2.3.6.

2.6. Minimal model program for surfaces

Let us present the approach of the MMP toward the birational classification of Algebraic Varieties for all dimension ≥ 2. (The case of smooth curves is clearly settled by the Riemann uniformization theorem.)

Consider a smooth projective variety X. The aim of Minimal Model Theory is to distinguish, inside the set of varieties which are birational to X, a special "minimal" member $\tilde{X}$ so to reduce the study of the birational geometry of X to that of $\tilde{X}$. The first basic fact is therefore to define what it means to be minimal. This is absolutely a non trivial problem and the following definition is the result of hard work of persons like Mori and Reid in the late 70's.

Definition 2.6.1. A variety $\tilde{X}$ is a **minimal model** if

- $\tilde{X}$ has $\mathbb{Q}$-factorial terminal singularities
- $K_{\tilde{X}}$ is nef

Let us make some observations on this definition. The second condition wants to express the fact that the minimal variety is *(semi) negatively curved*. We note in fact that if $detT_X = -K_X$ admits a metric with seminegative curvature then K_X is nef. The converse is actually an open problem (true in the case of surfaces and in general it may be considered as a conjecture).

The condition on the singularities is the real break-through of the definition. We will give some definitions in 3.3.4 but we refer the interested reader to [Re2]. The point of view should be the following: we are in principle interested in smooth varieties but we will find out that there are smooth varieties which does not admit smooth minimal models. However we can find such a model if we admit very mild singularities, the ones stated in the definition.

Remark 2.6.2. Terminal singularities are smooth points in the surface case (see 3.3.5). Thus a smooth surface X is a minimal model iff K_X is nef.

It happens that in the birational class of a given variety there is not a minimal model, think for instance of rational varieties. But the MMP hopes to make a list of those special varieties.

Given the definition of minimal variety we now want to show how, starting from X, one can determine a corresponding minimal model $\tilde{X}$. In view of 2.3.3 and of 2.5.7 the way to do it is quite natural. Namely, if K_X is not nef, then by 2.3.3 there exists an extremal ray (on which K_X is negative) and by 2.5.7 we can associate to it a contraction $f : X \to X'$ which contracts all curves in this ray into a normal projective variety X'.

A naive idea at this point would be the following.

If f is of fiber type, i.e. $dimX' < dimX$, then one hopes to recover a description of X via f. Indeed, by induction on the dimension, one should know a description of X' and of the fibers of f, which are, at least generically, Fano varieties. In fact in the surface case we have a precise description of these maps (see 2.5.7).

If f is birational then one thinks to substitute X with X' and proceed inductively.

The problem is of course that X' can be singular (now it starts to be clear that the choice of the singularities in the above definition is crucial). In general one can say just that it has normal singularities. However in the surface case the situation is optimal, namely Theorem 2.5.7 says that if $cont_R$ is birational then the image is again a smooth surface. Then apply recursively 2.5.7.1 and obtain that after finitely many blow downs of (-1)-curves one reaches a smooth surface S' with either $K_{S'}$ nef or with an extremal ray of fiber type. We stress again that in this case while performing the MMP we stay in the category of smooth surfaces. If $cont_R$ is of fiber type then, again by 2.5.7, its description is very precise. We have thus proved the following.

Theorem 2.6.3 (Minimal Model Program for surfaces)**.** *Let X be a smooth surface. After finitely many blow downs of (-1)-curves, $X \to X_1 \to ... \to X_n = X'$, one reaches a smooth surface X' satisfying one of the following:*

1) *$K_{X'}$ is nef i.e. X' is a minimal model*
2) *X' is a ruled surface*
3) *$X' \simeq \mathbb{P}^2$.*

Proposition 2.6.4. *If $K_{X'}$ is nef then the morphism $X \to X'$ is unique, thus X' is determined by X.*

Proof. Let $E \subset X$ be the exceptional curve of $p : X \to X'$. Since p is the composition of blow-ups of points we see that $K_X = p^*K_{X'} + F$ where F is an effective divisor and $SuppF = SuppE$.

Let $\varphi : X \to Y$ be a contraction as in 2.5.7 and $C \subset X$ be a curve such that $\varphi(C) = pt$. Then $K_X C < 0$. Since $K_{X'}$ is nef we obtain that

$$CF = CK_X - p_*CK_{X'} \le CK_X < 0.$$

Therefore $C \subset F$ and φ is a contraction as in 2.5.7 (1). Moreover there is a factorization (see the next lemma 3.2.3)

$$p : X \to Y \to X'.$$

The uniqueness of X' follows by induction on the number of exceptional curves of $X \to X'$.

Definition 2.6.5. A non singular projective surface X with a morphism $\varphi : X \to W$ is a **Mori fiber space** if φ is an extremal contraction of a ray and $dimX > dimW$.

Corollary 2.6.6. *Mori fiber spaces of dimension* 2 *are the ones described in 2) and 3) of the above theorem.*

Part 3. Birational theory for surfaces

3.1. Castelnuovo rationality criterium

We start deriving a famous Theorem from the Theorem 2.6.3.

Theorem 3.1.1. *Let X be a smooth projective surface.*

i) Let $p : \mathbb{P}^2 \dashrightarrow X$ be a dominant morphism. Then X is a rational surface.
ii) X is rational iff $h^1(\mathcal{O}_X) = h^0(\mathcal{O}_X(2K_X)) = 0$

Proof. If X is rational then the conditions in ii) are obviously satisfied.

All the conditions are birational invariant (see 3.2.5), so it is sufficient to consider the case when X satisfies one of the conditions of 2.6.3. If $X = \mathbb{P}^2$ then X is rational. Let X be a ruled surface over a curve C: in the case i) let $l \subset \mathbb{P}^2$ be a general line. The resulting map $p_l : l \dashrightarrow C$ is dominant, hence C is rational. In the case ii) $h^1(\mathcal{O}_C) = h^1(\mathcal{O}_X) = 0$, hence again C is rational. If $C = \mathbb{P}^1$ then X is rational (not immediate!).

Finally we show that K_X cannot be nef. In the case i) $deg p_l^* K_X \leq l K_{\mathbb{P}^2} = -3$. In the second case $h^0(\mathcal{O}_X(2K_X)) = 0$ thus also $h^0(\mathcal{O}_X(K_X)) = 0$. Therefore

$$\chi(\mathcal{O}_X) = 1 - h^0(\mathcal{O}_X) + h^0(\mathcal{O}_X(K_X)) = 1.$$

Also $h^2(\mathcal{O}_X(-K_X)) = h^0(\mathcal{O}_X(2K_X)) = 0$. If K_X is nef then $K_X^2 \geq 0$ (see 2.1.1), hence by RR

$$h^0(\mathcal{O}_X(-K_X)) \geq \chi(\mathcal{O}_X(-K_X)) = K_X^2 + 1.$$

Let $D \in |-K_X|$. $-D$ is nef, thus $D = \emptyset$. Therefore K_X is trivial and $h^0(\mathcal{O}_X(2K_X)) = 1$, a contradiction.

3.2. Factorization of birational morphisms

Theorem 3.2.1. *Let*

$$\varphi : X \to S$$

be a birational morphism between nonsingular projective surfaces X and S. Then φ is a composite of blowdowns, i.e. there is a sequence of contractions of (-1) curves,

$$\varphi_i : X_i \to X_{i+1},$$

for $i = 0, ..., l$, starting from $X_0 = X$ and ending with $X_{l+1} = S$.

Proof. Assume φ is not an isomorphism. Then $K_X - \varphi^* K_S = R$ is a non zero *effective* divisor, the **ramification divisor**. Since φ is birational R is coincide with the exceptional locus of φ and $\varphi(R)$ has codimension 2, i.e. it is a finite number of points (for further details see for instance the section 1.10 of [De]).

Lemma 3.2.2. *Let*

$$\varphi : X \to S$$

be a birational morphism between projective surfaces X and S and assume that X is smooth. Then for any non zero divisor $R \subset X$ such that $\varphi(R) = pt$ we have $R^2 < 0$.

Proof. The proof follows immediately from the Index theorem 1.3.3.

By the Lemma $R^2 < 0$ and hence there exists a curve $E \subset R$ such that

$$0 > ER = E(\varphi^* K_S + R) = EK_X \text{ and } E^2 < 0.$$

Thus E is a (-1) curve on X such that $\varphi(E) = pt$. Let $\nu : X \to X_1$ be the contraction of E given by the Castelnuovo contraction theorem 2.4.1. By the following lemma φ factors through ν, i.e. there exists $\varphi' : X_1 \to S$, a birational morphism, such that $\varphi = \varphi' \circ \nu$. If φ' is an isomorphism we are done, otherwise we repeat the argument. Since $\rho(X) = \rho(X_1) + 1$ the process has to come to an isomorphism after a finite number of steps.

Lemma 3.2.3. *Let $\varphi : X \to S$ and $\nu : X \to X_1$ surjective morphisms between normal projective surfaces such that $\varphi_*(\mathcal{O}_X) = \mathcal{O}_S$ and $\nu_*(\mathcal{O}_X) = \mathcal{O}_{X_1}$ (equivalently such that the morphisms have connected fibers). Suppose $\varphi(\nu^{-1}(p) = pt$ for all $p \in X_1$. Then there exist a unique morphism $\varphi' : X_1 \to S$ (with connected fibers) such that $\varphi = \varphi' \circ \nu$.*

Proof. The map φ' is defined set theoretically by $\varphi'(p) = \varphi(\nu^{-1}(p)$. It is well defined by the assumption and it is a continuous map in the Zariski topology, since φ and ν are proper.

For any open subset $U_S \subset S$ if we set $U_{X_1} = \varphi'^{-1}(U_S)$ then $\varphi^{-1}(U_S) = \nu^{-1}(U_{X_1})$. Therefore we have

$$\Gamma(U_S, \mathcal{O}_S) = \Gamma(\varphi^{-1}(U_S), \mathcal{O}_X) = \Gamma(\nu^{-1}(U_{X_1}), \mathcal{O}_X) = \Gamma(U_{X_1}), \mathcal{O}_{X_1})$$

which defines a map

$$\mathcal{O}_S \to \varphi'_*(\mathcal{O}_{X_1})$$

(actually an isomorphism). This gives to φ' the structure of a morphism from X_1 to S as local ringed space, i.e. a morphism as varieties. Uniqueness is immediate.

Corollary 3.2.4. *Let*

$$\varphi : X_1 \dashrightarrow X_2$$

be a birational morphism between nonsingular projective surfaces. Then there exists a sequence of blow-ups, $\psi_1 : V \to X_1$, followed by a sequence of blow-down, $\psi_2 : V \to X_2$, that factorize φ.

Proof. Take the graph $\Gamma \subset X_1 \times X_2$ and its desingularization $\tau : V \to \Gamma$; let $p_i : \Gamma \to X_i$ be the two projections.

Then by the previous theorem $p_i \circ \tau : V \to X_i$ factor through a sequence of blow-ups (or blow-down) and they give the desired factorization.

Corollary 3.2.5. *Let*

$$\varphi : X_1 \dashrightarrow X_2$$

be a birational morphism between nonsingular projective surfaces. Then there exists isomorphisms

$$H^i(X_1, \mathcal{O}_{X_1}) = H^i(X_2, \mathcal{O}_{X_2})$$

$$H^0(X_1, \mathcal{O}_{X_1}(mK_{X_1})) = H^0(X_2, \mathcal{O}_{X_2}(mK_{X_2})),$$

for any $m \in \mathbb{N}$, induced by φ.

Definition 3.2.6. Let X be a smooth projective surface. The **Kodaira dimension** , $k(X)$, of X is defined to be

$$k(X) = -\infty, \text{ if } H^0(X, \mathcal{O}_X(mK_X)) = 0 \text{ for all } m \in \mathbb{N}$$

$$k(X) = (\text{trasdeg}_{\mathbb{C}} \oplus_m H^0(X, \mathcal{O}_X(mK_X))) - 1,$$
$$\text{if } H^0(X, \mathcal{O}_X(mK_X)) \neq 0 \text{ for some } m \in \mathbb{N}.$$

Corollary 3.2.7. *The Kodaira dimension is a birational invariant.*

The following is the most important property of minimal model in dimension 2. We will not prove it here; the proof is not immediate and can be find for instance in section 1.5 of [Ma].

Theorem 3.2.8. *Let X be a smooth projective surface and assume that X is a minimal model (2.6.1 or 2.6.2). Then $k(X) \geq 0$, or equivalently $H^0(X, \mathcal{O}_X(mK_X)) \neq 0$ for some $m \in \mathbb{N}$.*

It follows immediately from the theorem and the previous section the following.

Corollary 3.2.9. *Let X be a smooth projective surface. Then an end result of the Minimal Model Program (MMP) for X (see 2.6.3) is a minimal model (respectively a Mori fiber space) iff $k(X) \geq 0$ (respectively $k(X) = -\infty$).*

On this line, with some extra work which we will not do here (see for instance section 1.5 of [Ma]), one can prove the following.

Theorem 3.2.10 (Abundance theorem)**.** *Let X be a smooth projective surface and assume that X is a minimal model. Then mK_X is base point free for sufficiently large $m \in \mathbb{N}$.*

3.3. Singularities and log singularities.

In this section we will briefly say something about singularities. Let X be a surface, that is a 2 dimensional algebraic variety. We do not assume that X is smooth but simply that it has normal singularities. This is equivalent to say that the singularities are isolated plus an algebraic condition on the stalk $\mathcal{O}_{X,x}$ (it has to be Cohen-Macaulay). In particular one can define (in many ways) a dualizing sheaf K_X.

A **resolution of** X is a birational projective morphism $f : Y \to X$ from a non singular projective surface. The following is an important and difficult theorem.

Theorem 3.3.1 (resolution of singularities: del Pezzo, ... , Hironaka)**.** *Given a normal projective surface a resolution always exists.*

If one have a curve in a smooth surface $C \subset X$ then it is much easier to resolve its singularities within X. Namely we have

Theorem 3.3.2 (embedded resolution)**.** *Let X be a smooth surface and $C \subset X$ a curve. Then there exists a finite sequence of blow ups, $X' \to X_1 \to ... \to X_n = X$, such that if $f : X' \to X$ is their composition then the total inverse image $f^{-1}(C)$ is a divisor with normal crossing (i.e. it is the connected union of a finite number of smooth curve in X' which intersect transversally).*

Proof. The proof is an easy exercise (see [Ha] p. 391) using 2.4.2.

Combining the above two results we can say the following

Corollary 3.3.3. *Let $C \subset X$ be a curve in a normal surface X. Then there exists a* **log resolution** *of the pair (X, C), that is a resolution $f : Y \to X$ such that $f^{-1}(C)$ is a divisor with normal crossing.*

Let X be a normal surface such that mK_X is a line bundle for some positive integer m; we say that X is $\mathbb{Q}$-Gorenstein. Let $C_j \subset X$ be curves and $D = \Sigma c_k C_k$, $c_k \in \mathbb{Q}$, an effective cycle. Let $f : Y \to X$ a log resolution of the pair $(X, \cup C_j)$ and E_i be all the exceptional curves in Y. Let $\tilde{D}$ be the strict transform of D. Then

$$K_Y + \tilde{D} \equiv f^*(K_X + D) + \Sigma a_i E_i.$$

Definition 3.3.4. The pair (X, D) is said to have terminal (respectively canonical, log terminal) singularities if $a_j > 0$ (resp. $a_j \geq 0$, $a_j > -1$), for any j.

The surface X is said to have terminal (respectively canonical, log terminal) singularities if the pair $(X, 0)$ has this type of singularities.

Proposition 3.3.5. *Let X be a normal surface and D be an effective cycle. Then*
i) X has terminal singularities if and only if it is smooth
ii) X has canonical singularities if and only if it has only rational double points (also called Du Val singularities)
iii) Assume that X is smooth. Then (X, D) does not have canonical singularities if and only if there exists a point $x \in X$ such that $mult_x D > 1$.

Proof. The proof of i) is an easy exercise using 3.2.2 and 2.4.1 and iii) is obvious. The point ii) is more delicate, but it can be taken as a definition of rational double point (use again 3.2.2).

3.4. Castelnuovo-Noether theorem and some definitions in the Sarkisov program

Sarkisov program is devoted to study the possible birational, not biregular, maps between Mori spaces.

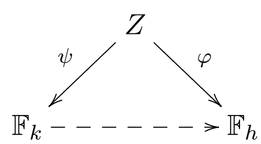
We do not want here to outline the complete program and its applications, for this we refer the reader to the recent book [Ma]. However we like to give an idea of its techniques and possible applications in the simpler set up of surfaces; for this, using Sarkisov dictionary, we prove the following beautiful Theorem.

Theorem 3.4.1 (Noether-Castelnuovo)**.** *The group of birational transformations of the projective plane is generated by linear transformations and the standard Cremona transformation, that is*

$$(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1),$$

where $(x_0 : x_1 : x_2)$ *are the coordinates of* $\mathbb{P}^2$*.*

Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational map which is not an isomorphism. To study the map χ we start *factoring* it with simpler birational maps, called *elementary links*, between Mori Spaces. More precisely an elementary link is either
i) the blow up of a point in $\mathbb{P}^2$ and its inverse, or
ii) an *elementary transformation* of a rational ruled surface $\mathbb{F}_k$ at a point $x \in \mathbb{F}_k$, that is



where ψ is the blow up of x and φ is the blow down of the strict transform of the fiber which contains the point x.

Remark 3.4.2. In ii) $h = k \pm 1$ according to the position of the point with respect to C_0, the section of $\mathbb{F}_k$ with $C_0^2 = -k$.

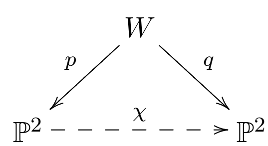
Consider $\mathcal{H} = \chi_*^{-1}\mathcal{O}(1)$, the strict transform of lines in $\mathbb{P}^2$; then $\mathcal{H}$ is without fixed components and $\mathcal{H} \subset |\mathcal{O}(n)|$ for some $n > 1$. Our point of view is to consider the general element $H \in \mathcal{H}$ as a twisted line. The factorisation we are aiming should "untwist" H step by step so to give back the original line hence the starting $\mathbb{P}^2$. Observe that the fact that χ is not biregular is encoded in the base locus of $\mathcal{H}$, therefore the untwisting is clearly related to the singularities of the pair $(\mathbb{P}^2, \mathcal{H})$, where by the pair $(\mathbb{P}^2, \mathcal{H})$ we understand the pair $(\mathbb{P}^2, H)$ where $H \in \mathcal{H}$ is a general element.

Theorem 3.4.3. *Let* $\mathcal{H} \subset |\mathcal{O}(n)|$ *be as above; then the pair* $(\mathbb{P}^2, (3/n)\mathcal{H})$ *has not canonical singularities.*

In particular there is a point $x \in \mathbb{P}^2$ *such that*

$$\operatorname{mult}_x \mathcal{H} > n/3. \tag{3.4.1}$$

Proof of Theorem 3.4.3. Take a resolution of χ



such that p is a log resolution of $(\mathbb{P}^2, \mathcal{H})$.

Pull back the divisor $K_{\mathbb{P}^2} + (3/n)\mathcal{H}$ and $K_{\mathbb{P}^2} + (3/n)\mathcal{O}(1)$ via p and q respectively. If $\mathcal{H}_W$ is the strict transform of H we have

$$\begin{aligned} K_W + (3/n)\mathcal{H}_W &= \\ p^*\mathcal{O}_{\mathbb{P}^2} + \sum_i a_i' E_i + \sum_h c_h G_h &= q^*\mathcal{O}_{\mathbb{P}^2}(3(1/n-1)) + \sum_i a_i E_i + \sum_j b_j F_j \end{aligned}$$

where E_i are p and q exceptional divisors, while F_j are q but not p exceptional divisors and G_h are p but not q exceptional divisors. (Observe that, since $\mathcal{O}(1)$ is base point free, the a_i's and b_j's are positive integers).

Let $l \subset \mathbb{P}^2$ a general line in the right hand side plane. In particular q is an isomorphism on l and therefore $E_i \cdot q^*l = F_j \cdot q^*l = 0$ for all i and j.

The crucial point is that on the right hand side we have some negativity coming from the non effective divisor $K_{\mathbb{P}^2} + (3/n)\mathcal{O}(1)$ that has to be compensated by some non effective exceptional divisor on the other side.

More precisely, since $n > 1$, we have on one hand that

$$(K_W + (3/n)\mathcal{H}_W) \cdot q^*l = (q^*\mathcal{O}_{\mathbb{P}^2}(3(1/n-1)) + \sum_i a_i E_i + \sum_j b_j F_j) \cdot q^*l < 0,$$

and on the other hand that

$$0 > (K_W + (3/n)\mathcal{H}_W) \cdot q^*l = (p^*\mathcal{O}_{\mathbb{P}^2} + \sum_i a_i' E_i + \sum_h c_h G_h) \cdot q^*l.$$

So that $c_h < 0$ for some h, that is $(\mathbb{P}^2, (3/n)\mathcal{H})$ is not canonical. □

The above proof can be generalised to the following set up. Let $\pi : X \to S$ and $\varphi : Y \to W$ be two Mori spaces. Let $\chi : X \dashrightarrow Y$ a birational not biregular map. Choose $\mathcal{H}_Y$ a very ample linear system on Y. Let $\mathcal{H} = \chi_*^{-1}\mathcal{H}_Y$ then by the definition of Mori space there exists a $\mu \in \mathbb{Q}$ such that $K_X + (1/\mu)\mathcal{H} \equiv_\pi 0$.

Theorem 3.4.4 (Noether–Fano inequalities, see [Ma] Prop. 1.8.9.)**.** *In the above notation, in particular with χ non biregular and $K_X + (1/\mu)\mathcal{H} \equiv_\pi 0$, then either $(X, (1/\mu)\mathcal{H})$ has not canonical singularities or $K_X + (1/\mu)\mathcal{H}$ is not nef.*

We are now ready to start the factorisation of χ.

For this let $x \in \mathbb{P}^2$ be a point such that $(\mathbb{P}^2, (3/n)\mathcal{H})$ is not canonical at x and let $\nu : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow up of x, with exceptional divisor C_0. In the context of Sarkisov theory this is called a *terminal extraction* and this will be our first elementary link .

Let $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ be the Mori space structure of $\mathbb{F}_1$. Let $\chi' = \chi \circ \nu : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$ and $\mathcal{H}' = (\chi')_*^{-1}\mathcal{O}(1)$. Let $n' = n - mult_x\mathcal{H}$ and consider the pair $(\mathbb{F}_1, (2/n')\mathcal{H}')$.

Let us first notice that if $f \subset \mathbb{F}_1$ a generic fiber of the ruled structure then

$$K_{\mathbb{F}_1} + (2/n')\mathcal{H}' \cdot f = 0,$$

thus

$$K_{\mathbb{F}_1} + (2/n')\mathcal{H}' \equiv_{\pi_1} 0.$$

On the other hand

$$\begin{aligned}(K_{\mathbb{F}_1} + (2/n')\mathcal{H}')\cdot C_0 &= -1 + (2/n')mult_x\mathcal{H}\\ &= (-n + 3mult_x\mathcal{H})/n' > 0,\end{aligned}$$

the last inequality coming from 3.4.3.

Thus $K_{\mathbb{F}_1} + (2/n')\mathcal{H}'$ is nef. and we are in the conditions to apply Theorem 3.4.4 to conclude that the pair $(\mathbb{F}_1, (2/n')\mathcal{H}')$ is not canonical and therefore the linear system $\mathcal{H}'$ admits a point $x' \in \mathbb{F}_1$ with multiplicity greater than $n'/2 = \frac{\mathcal{H}'\cdot f}{2}$.

The next step is an elementary transformation at x'

(3.4.2)
$$\begin{array}{ccc} & Z & \\ {\scriptstyle\psi}\swarrow & & \searrow{\scriptstyle\varphi} \\ \mathbb{F}_1 & \dashrightarrow & \mathbb{F}_h \end{array}$$

Let $x_2 \subset \mathbb{F}_h$ be the exceptional locus of φ^{-1} and $\mathcal{H}_2$ be the strict transform of $\mathcal{H}'$. Observe the following two facts:

i) $(K_{\mathbb{F}_h} + (2/n')\mathcal{H}_2)\cdot f = 0$, where, by abuse of notation, f is the strict transform of $f \subset \mathbb{F}_1$,

ii) since $mult_{x'}\mathcal{H}' > \frac{\mathcal{H}'\cdot f}{2}$, then $(\mathbb{F}_h, (2/n')\mathcal{H}_2)$ has terminal singularities at x_2.

In particular i) says that we can apply again Theorem 3.4.4 to the log pair $((\mathbb{F}_h, (2/n')\mathcal{H}_2)$ while ii) says that we did not introduce any new canonical singularities since the point x_2 is a terminal singularity for this pair.

Therefore after finitely many elementary transformations we "delete" all non canonical singularities and we reach a pair $(\mathbb{F}_k, (2/n')\mathcal{H}_r)$ with canonical singularities such that

$$K_{\mathbb{F}_k} + (2/n')\mathcal{H}_r \equiv_{\pi_k} 0.$$

Then, again by Theorem 3.4.4, the pair $(\mathbb{F}_k, (2/n')\mathcal{H}_r)$ cannot be nef.

Observe that $NE(\mathbb{F}_k)$ is a two dimensional cone. In particular it has only two rays. One is spanned by f, a fiber of π_k. Let Z an effective irreducible curve in the other ray. Then

$$(K_{\mathbb{F}_k} + (2/n')\mathcal{H}_r)\cdot Z < 0. \tag{3.4.3}$$

Since $\mathcal{H}_r$ has not fixed components then $\mathbb{F}_k$ is a del Pezzo surface.

It is easy to prove that if $\mathbb{F}_k$ is a del Pezzo surface then the only possibilities are $k = 0, 1$.

Now in case $k = 1$ we simply blow down the exceptional curve $\nu : \mathbb{F}_1 \to \mathbb{P}^2$, and reach $\mathbb{P}^2$ together with a linear system $\nu_*\mathcal{H}_2 =: \tilde{\mathcal{H}} \subset |\mathcal{O}(j)|$. Note that in this case, by equation (3.4.3),

$$K_{\mathbb{F}_1} + (2/n')\mathcal{H}_r = \nu^*(K_{\mathbb{P}^2} + (2/n')\tilde{\mathcal{H}}) + \delta C_0,$$

for some positive δ. Therefore $K_{\mathbb{P}^2} + (2/n')\tilde{\mathcal{H}}$ is not nef (since $0 = (K_{\mathbb{F}_1} + (2/n')\mathcal{H}_r)f = (\nu^*(K_{\mathbb{P}^2} + (2/n')\tilde{\mathcal{H}}))f + (\delta C_0)f = (\nu^*(K_{\mathbb{P}^2} + (2/n')\tilde{\mathcal{H}}))f + \delta)$. In other terms

$$(2/n')j < 3,$$

and

$$j < \frac{3(n - mult_x\mathcal{H})}{2} < n.$$

Now we iterate the above argument, i.e. we restart at the beginning of the proof but with the pair $(\mathbb{P}^2, (3/j)\tilde{\mathcal{H}})$; the above strict inequality $j < n$, tells us that and after finitely many steps we untwist the map χ, i.e. we reach $\mathbb{P}^2$ with a linear system $\mathcal{H} = |\mathcal{O}(1)|$.

In case $k = 0$ observe that $\mathbb{F}_0 \simeq \mathbb{Q}^2$ is a Mori space for two different fibrations, let f_0 and f_1 the general fibers of these two fibrations. Moreover by equation (3.4.3)

$$(K_{\mathbb{F}_0} + (2/n')\mathcal{H}_r) \cdot f_1 < 0.$$

That is there exists an

$$n_1 < n' \tag{3.4.4}$$

such that

$$(K_{\mathbb{F}_0} + (2/n_1)\mathcal{H}_r) \cdot f_0 > 0,$$

and

$$(K_{\mathbb{F}_0} + (2/n_1)\mathcal{H}_r) \cdot f_1 = 0.$$

Again by Theorem 3.4.4, this time applied with respect to the fibration with fiber f_1, this implies that $(\mathbb{F}_0, (2/n_1)\mathcal{H}_r)$ is not canonical and we iterate the procedure. As in the previous case the strict inequality of equation (3.4.4) implies a termination after finitely many steps.

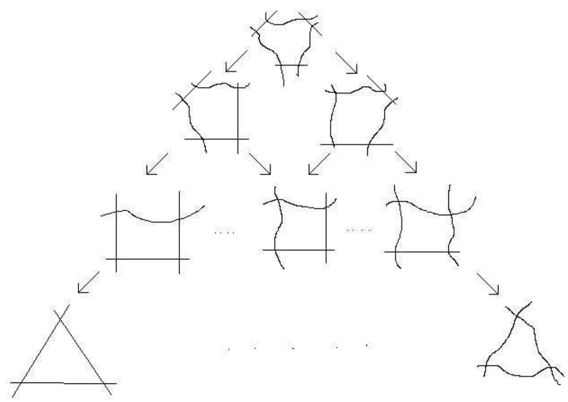
Thus we have factorised any birational, not biregular, self-map of $\mathbb{P}^2$ with a sequence of "elementary links", namely elementary transformations and blow ups of $\mathbb{P}^2$ at a point.

The next step is to interpret a standard Cremona transformation in this new language, i.e. in term of the elementary links we have introduced above.

Exercise 3.4.5. *Prove that a standard Cremona transformation is given by the following links*

$$\begin{array}{ccccc} & \mathbb{F}_1 & \dashrightarrow \mathbb{F}_0 \dashrightarrow & \mathbb{F}_1 & \\ \swarrow & & & & \searrow \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \end{array}$$

Proof. The untwisting of the Cremona transformation is illustrated by the following picture.



Proof of Theorem 3.4.1. Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational map and

(3.4.5)
$$\begin{array}{ccc} & \mathbb{F}_1 \overset{l_0}{\dashrightarrow} \mathbb{F}_k \overset{l_1}{\dashrightarrow} \cdots \dashrightarrow \mathbb{F}_1 & \\ \swarrow^{\nu_1} & & \searrow \\ \mathbb{P}^2 & & \mathbb{P}^2 \end{array}$$

the factorisation in elementary links obtained above. Let us first make the following observation. If there is a link leading to an $\mathbb{F}_1$ then we can break the birational map simply blowing down the (-1)-curve. That is substitute χ with the following two pieces

$$\begin{array}{ccccc} & \mathbb{F}_1 \overset{l_0}{\dashrightarrow} \cdots \overset{l_i}{\dashrightarrow} \mathbb{F}_1 & \overset{\sim}{\longrightarrow} & \mathbb{F}_1 \overset{l_{i+1}}{\dashrightarrow} \cdots \dashrightarrow \mathbb{F}_1 & \\ \swarrow^{\nu_1} & \searrow^{\nu_2} & & \swarrow^{\nu_2} & \searrow \\ \mathbb{P}^2 & \overset{\chi_1}{\dashrightarrow} & \mathbb{P}^2 & \overset{\chi_2}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

So that we can assume

(3.4.6) there are no links leading to $\mathbb{F}_1$ "inside" the factorisation.

Let

$$d(\chi) = \max\{k : \text{there is an } F_k \text{ in the factorisation}\}.$$

We prove the Theorem by induction on $d(\chi)$. If $d(\chi) = 1$ then χ is a Cremona transformation by 3.4.5.

If $d(\chi) \geq 2$, by assumption (3.4.6), then l_0 is of type $\mathbb{F}_1 \dashrightarrow \mathbb{F}_2$ and l_1 is of type $\mathbb{F}_2 \dashrightarrow \mathbb{F}_k$. Then we force Cremona like diagrams in it, at the cost of introducing new singularities. Let

$$\begin{array}{ccccc} & \mathbb{F}_1 \overset{\alpha}{\dashrightarrow} \mathbb{F}_0 \overset{l_0}{\dashrightarrow} \mathbb{F}_1 & & \mathbb{F}_1 \overset{\alpha^{-1}}{\dashrightarrow} \mathbb{F}_0 \overset{l_1}{\dashrightarrow} \ldots \mathbb{F}_1 & \\ \swarrow^{\nu_1} & \searrow^{\nu_2} & & \swarrow^{\nu_2} & \searrow \\ \mathbb{P}^2 & & \mathbb{P}^2 & \overset{\chi'}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where $\alpha : \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$ is an elementary transformation centered at a general point of $\mathbb{F}_1$, and $Exc(\alpha^{-1}) = \{y_0\}$. So that $\alpha_*(\mathcal{H}')$ has an ordinary singularity at y_0. Then l_0 is exactly the same modification but leads to an $\mathbb{F}_1$ and ν_2 is the blow down of the exceptional curve of this $\mathbb{F}_1$. Observe that neither α nor ν_2 are links in the Sarkisov category, in general. Nonetheless the first part can be factorised by standard Cremona transformations. Let $\chi' = \chi_1 \circ \ldots \circ \chi_k$ a decomposition of χ in pieces satisfying (3.4.6). Then $d(\chi_i) < d(\chi)$ for all $i = 1, \ldots, k$. Therefore by induction hypothesis also χ' can be factorised by Cremona transformations. Hence χ is factorised by Cremona transformations □

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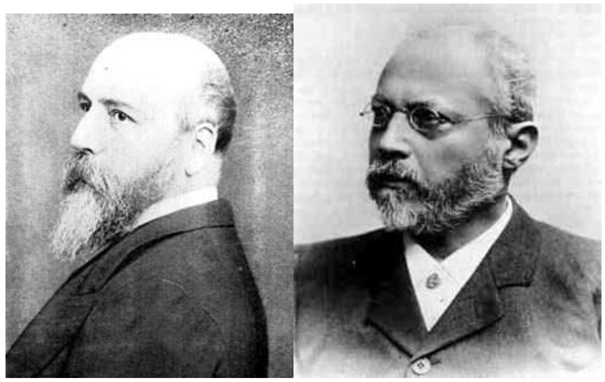
Some of the Main Actors



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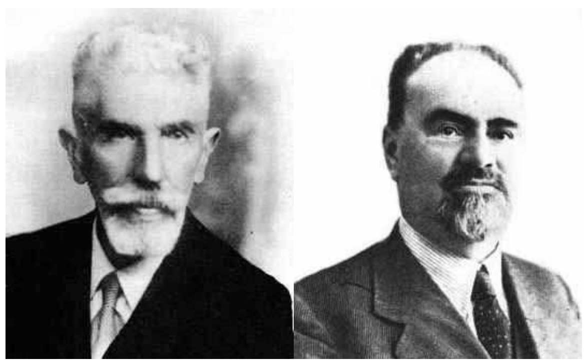
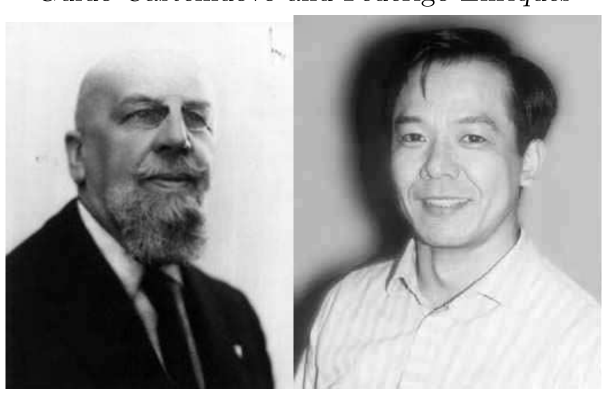


Guido Castelnuovo and Federigo Enriques



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