4-DIMENSIONAL SYMPLECTIC CONTRACTIONS

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Abstract. Four dimensional symplectic resolutions are (relative) Mori Dream Spaces. Any two such resolutions are connected by a sequence of Mukai flops. We discuss cones of their movable divisors with faces determined by curves whose loci are divisors, we call them essential curves. These cones are divided into nef chambers related to different resolutions, the division is determined by classes of flopping 1-cycles. We also study schemes parametrizing minimal essential curves and show that they are resolutions, possibly non-minimal, of surface Du Val singularities.

1. Introduction

In the paper we consider *local symplectic contractions* of 4-folds. That is, we deal with maps $\pi : X \to Y$ where

- X is a smooth complex 4-fold with a closed holomorphic 2-form, non-degenerate at every point,
- Y is an affine (or Stein) normal variety,
- π is a birational projective morphism.

In dimension 2 symplectic contractions are classical and they are minimal resolutions of Du Val singularities. In fact, any symplectic contraction can be viewed as a special symplectic resolution of a symplectic normal singularity.

General properties of symplectic contractions (in arbitrary dimension) have been considered in a number of papers published in the last decade: [Bea00], [Ver00], [Nam01], [Kal02], [Kal03], [Wie03], [FN04], [GK04], [Fu06a], [HT09], [Bel09], [LS08], to mention just a few; see also [Fu06b] for more references and a review on earlier developments in this subject. Let us just recall two beautiful results about symplectic contractions: these maps are semismall, [Wie03], and McKay correspondence holds for those symplectic contractions which are resolutions of quotient symplectic singularities, [Kal02], [GK04]. However, in dimension 4 and higher, apart of the desccription in codimension 2, [Wie03], not much is known about the fine geometrical structure of these morphisms which is the problem we want to tackle in the present paper.

The 4-dimensional small case (i.e. when π does not contract a divisor) is known by [WW03, Thm. 1.1]. Using this result we first prove in section 3.1 that X is a Mori Dream Space over Y, as defined in [HK00]. In short, every movable divisor of X (over Y) can be made nef and semiample after a finite number of small $\mathbb{Q}$-factorial modifications (flops), see also [WW03, Thm. 1.2] where a version of this result was

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1991 *Mathematics Subject Classification.* 20C10, 14E15, 14F10, 14J28, 14J32.

The first author was supported by the Italian PRIN; he thanks also the Institute of Mathematics of Warsaw University for hospitality. The second author was supported by Polish MNiSzW grants N N201 2653 33 and N N201 4206 39 as well as Fondazione Bruno Kessler at CIRM in Trento. Thanks to Andrzej Weber for stimulating discussions.

announced. We describe the cone of movable divisors, 4.1, and we give properties of its subdivision into chambers corresponding to nef cones of small $\mathbb{Q}$-factorial modifications (SQM's) of X, 4.5.
In section 5, following the approach introduced in [Wie03] and subsequently in [SCW04], we study families of rational curves (i.e. irreducible components of the Chow scheme) in X/Y. They are resolutions of Du Val singularities, 5.1, possibly non-minimal with discrepancy depending on the rank of the evaluation map, 5.2, and depending on the SQM model, 6.5. We also show that studying 4 dimensional symplectic resolutions implies understanding arbitrary dimensional case in codimension 4, via the argument of general intersection of a suitable number of divisors, or vertical slicing, 5.4.
In section 6 we study known examples of resolutions of quotient symplectic singularities and we describe explicitly their movable cones and families of rational curves on them. In particular, we describe explicitly the division of the movable cone of a symplectic resolution of $\mathbb{C}^4/(\mathbb{Z}_{n+1} \wr \mathbb{Z}_2)$ into nef cones associated to different resolutions of this singularity.

2. Notation and preliminaries

2.1. **Symplectic contractions.** A holomorphic 2-form ω on a smooth variety is called **symplectic** if it is closed and non-degenerate at every point. A **symplectic variety** is a normal variety Y whose smooth part admits a holomorphic symplectic form ω_Y such that its pull back to any resolution $\pi : X \to Y$ extends to a holomorphic 2-form ω_X on X. We call π a **symplectic resolution** if ω_X is non degenerate on X, i.e. it is a symplectic form. More generally, a map $\pi : X \to Y$ is called a **symplectic contraction** if X is a symplectic manifold, Y is normal and π is a birational projective morphism. If moreover Y is affine we will call $\pi : X \to Y$ a **local symplectic contraction** or **local symplectic resolution**. The following facts are well known, see the survey paper [Fu06b].

Proposition 2.1. *Let Y be a symplectic variety and $\pi : X \to Y$ be a resolution. Then the following statement are equivalent: (i) $\pi^* K_Y = K_X$, (ii) π is symplectic, (iii) K_X is trivial, (iv) for any symplectic form on Y_{reg} its pull-back extends to a symplectic form on X.*

Theorem 2.2. *A symplectic resolution $\pi : X \to Y$ is semismall, that is for every closed subvariety $Z \subset X$ we have $2 \operatorname{codim} Z \geq \operatorname{codim} \pi(Z)$. If equality holds Z then is called a maximal cycle.*

Example 2.3. Let S be a smooth surface (proper or not). Denote by $S^{(n)}$ the *symmetric product* of S, that is $S^{(n)} = S^n/\sigma_n$, where σ_n is the symmetric group of n elements. Let also $Hilb^n(S)$ be the *Hilbert scheme* of 0-cycles of degree n. A classical result (c.f. [Fog68]) says that $Hilb^n(S)$ is smooth and that $\tau : Hilb^n(S) \to S^{(n)}$ is a crepant resolution of singularities. We will call it a Hilb-Chow map.
Suppose now that $S \to S'$ is a resolution of a Du Val singularity which is of type $S' = \mathbb{C}^2/H$ with $H < SL(2, \mathbb{C})$ a finite group. Then the composition $Hilb^n(S) \to S^{(n)} \to (S')^{(n)}$ is a local symplectic contraction.
We note that $(S')^{(n)}$ is a quotient singularity with respect to the action of the wreath product $H \wr \sigma_n = (H^n) \rtimes \sigma_n$.

2.2. **Mori Dream Spaces.** Let us recall basic definitions regarding Mori Dream Spaces. For more information we refer to [HK00] or [ADHL10]. Our definitions are far from general but sufficient for our particular local set-up. We assume that

(1) $\pi : X \to Y$ is a projective morphism of normal varieties with connected fibers, that is $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and $Y = \operatorname{Spec} A$ is affine,
(2) X is localy factorial and $\operatorname{Pic}(X/Y) = \operatorname{Cl}(X/Y)$ is a lattice (finitely generated abelian group with no torsion) so that $N^1(X/Y) = \operatorname{Pic}(X/Y) \otimes \mathbb{Q}$ is a finite dimensional vector space

By $\operatorname{Nef}(X/Y) \subset N^1(X/Y)$ we understand the closure of the cone spanned by the classes of relatively-ample bundles while by $\operatorname{Mov}(X/Y) \subset N^1(X/Y)$ we understand the cone spanned by the classes of linear systems which have no fixed components. That is, a class of a $\mathbb{Q}$-divisor D is in $\operatorname{Mov}(X/Y)$ if the linear system $|mD|$ has no fixed component for $m \gg 0$. The following is a version of [HK00, Def. 1.10].

Definition 2.4. In the above situation we say that X is a Mori Dream Space (MDS) over Y if in addition

(1) $\operatorname{Nef}(X/Y)$ is the affine hull of finitely many semi-ample line bundles:
(2) there is a finite collection of small $\mathbb{Q}$-factorial modifications (SQM) over Y, $f_i : X - \to X_i$ such that $X_i \longrightarrow Y$ satisfies the above assumptions and $\operatorname{Mov}(X/Y)$ is the union of the strict transforms $f_i^*(\operatorname{Nef}(X_i))$

We note that a version of [HK00, Prop. 2.9] works in the relative situation too. In particular, the relative Cox ring, $\operatorname{Cox}(X/Y)$, is a well defined, finitely generated, graded module $\bigoplus_{L\in Pic(X/Y)} \Gamma(X, L)$. Moreover X is a GIT quotient of $\operatorname{Spec}(\bigoplus_{L\in Pic(X/Y)} \Gamma(X, L))$ under the Picard torus $\operatorname{Pic}(Y/X) \otimes \mathbb{C}^*$ action.

Example 2.5. Take $\mathbb{C}^*$ action on $\mathbb{C}^r \times \mathbb{C}^r$ with coordinates (x_i, y_j) and weights 1 for x_i's and -1 for y_j's. Using these weights we define a $\mathbb{Z}$-grading of the polynomial ring and write $\mathbb{C}[x_i, y_j] = \bigoplus_{m\in\mathbb{Z}} A_m$. The quotient $\widehat{Y} = \operatorname{Spec} A_0$ is a toric singularity which, in the language of toric geometry, is associated to a cone spanned by r vectors e_i and f_j in the lattice of rank $2r - 1$, with one relation $\sum e_i = \sum f_j$. The result is the cone over Segre embedding of $\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$. Consider $A_+ = \bigoplus_{m\geq 0} A_m$ and $A_- = \bigoplus_{m\leq 0} A_m$, and define two varieties over $\widehat{Y}$:

$$\widehat{X}_\pm = \operatorname{Proj}_{A_0} A_\pm \longrightarrow \widehat{Y}$$

Both, X_+ and X_-, are smooth because, as toric varieties, they are ssociated to two unimodular triangulations of the cone in question: one in which we omit consecutive e_i's, the other in which we omit f_j's. The affine pieces of covering are of type $\operatorname{Spec} \mathbb{C}[x_i/x_k, x_k y_j]$, where $k = 1, \ldots, r$, for $\widehat{X}_+$ and similar for $\widehat{X}_-$. The two resolution $\widehat{X}_+ \to \widehat{Y} \leftarrow \widehat{X}_-$ form two sides of so-called Atiyah flop.

Consider an ideal $I = (\sum_i x_i y_i) \lhd \mathbb{C}[x_i, y_j]$ generated by a $\mathbb{C}^*$ invariant function (degree 0) and its respective counterparts $I_0 \cap A_0 \lhd A_0$, $I_+ \lhd A_+$ and $I_- \lhd A_-$. We set $Y = \operatorname{Spec} A_0/I_0$, $X_+ = \operatorname{Proj}_{A_0} A_+/I_+$ and $X_- = \operatorname{Proj}_{A_0} A_-/I_-$ and call the resulting diagram $X_+ \to Y \leftarrow X_-$ **Mukai flop**. The variety Y is symplectic since the form $\omega = \sum_i (dx_i \wedge dy_i)$ on $\mathbb{C}^r \times \mathbb{C}^r$ descends to a symplectic form on Y. The varieties $X_\pm$ are its small symplectic resolutions and $\mathbb{C}[x_i, y_j]/I$ is their Cox ring. We note that $\operatorname{Spec}(\mathbb{C}[x_i, y_j]/I)$ is the cone over the incidence variety of points and hyperplanes in $\mathbb{P}^{r-1} \times (\mathbb{P}^{r-1})^*$. Finally, we note that the movable cone $\operatorname{Mov}(X_\pm)$ is the whole line $N^1(X_\pm)$ hence it is is not strictly convex.

3. Local symplectic contractions in dimension 4.

3.1. **MDS structure.** In this section $\pi : X \to Y$ is a local symplectic contraction, as defined in 2.1 and $\dim X = 4$. By the semismall property, the fibers of π have dimension less or equal to 2 and the general non trivial fibers have dimension 1. We will denote with 0 the unique (up to shrinking Y into a smaller affine set) point such that $\dim \pi^{-1}(0) = 2$. We start by recalling the following theorem from [WW03, Thm. 1.1].

Theorem 3.1. *Suppose that π is small (i.e. it does not contract any divisor). Then π is locally analytically isomorphic to the collapsing of the zero section in the cotangent bundle of $\mathbb{P}^2$. Therefore X admits a Mukai flop as described in example 2.5*

The above theorem, together with Matsuki's termination of 4-dimensional flops, see [Mat91], is the key ingredient in the proof of the following result. See also [WW03, Thm.1.2] and [Wie02], as well as [BHL03]. The classical references for the Minimal Model Program (MMP), which is the framework for this argument, are [KMM87] and [KM98].

Theorem 3.2. *Let $\pi : X \to Y$ be a 4-dimensional local symplectic contraction and let $\pi^{-1}(0)$ be its only 2-dimensional fiber. Then X is a Mori Dream Space over Y. Moreover any SQM model of X over Y is smooth and any two of them are connected by a finite sequence of Mukai flops whose centers are over $0 \in Y$. In particular, there are only finitely many non isomorphic (local) symplectic resolution of Y.*

Proof. Firstly, by the Kawamata non-vanishing, the linear and numerical equivalence over Y are the same hence $\mathrm{Pic}(X/Y)$ is a lattice. By the Kawamata-Shokurov base point free theorem every nef divisor on X is also semiample. On the other hand, the rationality theorem asserts that $\mathrm{Nef}(X/Y))$ is locally rational polyhedral. Next we claim that X satisfies second property of definition 2.4: for this take a movable divisor and assume that it is not nef. Look for extremal rays which have negative intersection with it. They have to be associated to small contractions because they have to be in the base point locus of the divisor. By the theorem 3.1 these are contractions of a $\mathbb{P}^2$ which can be flopped (Mukai flop) so that the result remains smooth. The process has to finish by the result on the termination of flops (relative to the chosen movable divisor) by [Mat91]. Therefore, after a finite number of flops, the strict transform of the movable divisor in question becomes a nef divisor, which is semiample (by the base point free theorem). □

3.2. **Essential curves.** The following definition of essential curves is a simplified version of the one introduced in [AW10], suitable for the present set-up.

Definition 3.3. Let $\pi : X \to Y$ be a 4-dimensional local symplectic contraction with the unique 2-dimensional fiber $\pi^{-1}(0)$. By $N_1(X/Y)$ we denote the $\mathbb{Q}$ vector space of 1-cycles proper over Y. We define $\mathrm{Ess}(X/Y)$ as the convex cone spanned by the classes of curves which are not contained in $\pi^{-1}(0)$. Classes of curves in $\mathrm{Ess}(X/Y)$ we call *essential curves*.

Theorem 3.4. (c.f. [AW10]) *The cones* $\mathrm{Mov}(X/Y)$ *and* $\mathrm{Ess}(X/Y)$ *are dual in terms of the intersection product of* $N^1(X/Y)$ *and* $N_1(X/Y)$*, that is* $\mathrm{Mov}(X/Y) = \mathrm{Ess}(X/Y)^\vee$.

Proof. First we note that if D is a movable divisor on X, or if $|mD|$ has no fixed component for $m \gg 0$, then the base point locus of $|mD|$ is contained in $\pi^{-1}(0)$. This yields the obvious inclusion $\mathrm{Mov}(X/Y) \subseteq \mathrm{Ess}(X/Y)^{\vee}$. Moreover, since by 3.2 any two symplectic resolutions of Y are connected by a sequence of flops in centers over $0 \in Y$, it follows that the intersection of divisors with curves outside $\pi^{-1}(0)$ does not depend on the choice of the resolution (or SQM of X). That is, if $C \subset X \setminus \pi^{-1}(0)$ is a curve proper over Y and $\pi' : X' \to Y$ another symplectic resolution with D', the strict transform then $D \cdot C = D' \cdot C$.
Now assume by contradiction that $\mathrm{Mov}(X/Y) \neq \mathrm{Ess}(X/Y)^{\vee}$. Let F be a facet (codimension 1 face) of $\mathrm{Mov}(X/Y)$. Since X is MDS, we can take a resolution $\pi' : X' \to Y$, for which $F \cap \mathrm{Nef}(X'/Y)$ is an extremal face of $\mathrm{Nef}(X'/Y)$. The relative elementary contraction $X' \to Y' \to Y$ of the face $F \cap \mathrm{Nef}(X'/Y)$ is divisorial. Indeed, if it is not, then after a flop we would get another $X'' \to Y$ whose nef cone $\mathrm{Nef}(X''/Y)$ is on the other side of the face $F \cap \mathrm{Nef}(X'/Y)$, contradicting the fact that F is an extremal face of $\mathrm{Mov}(X/Y)$. Now we can choose a curve $C \subset X' \setminus (\pi')^{-1}(0) = X \setminus \pi^{-1}(0)$ contracted by $X' \to Y'$ and we get the inclusion $F \subset C^{\perp}$. Thus every facet of $\mathrm{Mov}(X/Y)$ is supported by a curve in $\mathrm{Ess}(X/Y)$, hence $\mathrm{Mov}(X/Y)^{\vee} \subseteq \mathrm{Ess}(X/Y)$ and we are done. □

From the proof it follows that the above result remains true also if $\pi : X \to Y$ is a higher dimensional symplectic contraction and X is MDS over Y, and essential curves are defined as those whose loci is in codimension 1.
We note that since the map π is assumed to be projective the cone $\mathrm{Nef}(X/Y)$, hence the cone $\mathrm{Mov}(X/Y)$, is of maximal dimension. On the other hand we have the following observation.

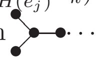
Proposition 3.5. *Let $\pi : X \to Y$ be as in 3.2 (or, more generally, suppose that X is MDS over Y).*
The following conditions are equivalent.

1. *the cone* $\mathrm{Ess}(X/Y)$ *is of maximal dimension,*
2. *the cone* $\mathrm{Mov}(X/Y)$ *is strictly convex, that is it contains no linear subspace of positive dimension,*
3. *the classes of components of fibers of π outside $\pi^{-1}(0)$ generate $N_1(X/Y)$,*
4. *the classes of exceptional divisors generate $N^1(X/Y)$.*

Proof. In view of 3.4 the equivalence of (1) and (2) is formal. Also (1) is equivalent to (3) by the definition of the cone $\mathrm{Ess}(X/Y)$. Finally, the intersection of classes of exceptional divisors with curves contained in general fibers of their contraction is a non-degenerate pairing, c.f. 4.1. Hence (3) is equivalent to (4). □

4. Root systems and the structure of $\mathrm{Mov}(X/Y)$.

4.1. **Root systems.** This is to recall generalities regarding root systems. A standard reference for this part is [Bou75]. We consider a (finite dimensional) real vector space V with a euclidean product and root lattice Λ_R and weight lattice $\Lambda_W \supset \Lambda_R$. We distinguish the set of simple (positive) roots denoted by $\{e_i\}$ and their opposite $E_i = -e_i$. Note that the lattice Λ_R is spanned by E_i's or e_i's while its $\mathbb{Z}$-dual is Λ_W. The Cartan matrix describes the intersection $(e_i \cdot e_j) = -(e_i \cdot E_j)$ which is also reflected in the respective Dynkin diagram. Any such root system is a (direct) sum of irreducible ones coming from the infinite series $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$ and also $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ as well as $\mathbb{F}_4$ and $\mathbb{G}_2$.

The Cartan matrix of each of the systems $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ has 2 at the diagonal and 0 or -1 outside the diagonal. Given a group H of automorphisms of any of the $\mathbb{A}-\mathbb{D}-\mathbb{E}$ Dynkin diagrams we can produce a matrix of intersections of classes of orbits of the action. The entries are intersections of an alement of the orbit with the sum of all elements in the orbit, that is: $(e_i \cdot \sum_{e_k \in H(e_j)} e_k)$. For example: the involution identifying two short legs of the $\mathbb{D}_n$ diagram  described by the $n \times n$ Cartan matrix

$$\begin{pmatrix} 2 & 0 & -1 & 0 & \cdots \\ 0 & 2 & -1 & 0 & \cdots \\ -1 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix}$$

yields the $(n-1)\times(n-1)$ matrix associated to the system $\mathbb{C}_{n-1}$:

$$\begin{pmatrix} 2 & -1 & 0 & \cdots \\ -2 & 2 & -1 & \cdots \\ 0 & -1 & 2 & \cdots \\ \cdots & \cdots & \cdots & \end{pmatrix}$$

If, by abuse, we denote by the respective letter the Cartan matrix associated to the appropriate root system and the quotient denotes the matrix of intersections of classes under the group action, then we verify that $\mathbb{A}_{2n+1}/\mathbb{Z}_2 = \mathbb{B}_n$, $\mathbb{D}_n/\mathbb{Z}_2 = \mathbb{C}_{n-1}$, $\mathbb{E}_6/\mathbb{Z}_2 = \mathbb{F}_4$ and $\mathbb{D}_4/\sigma_3 = \mathbb{G}_2$. The geometry behind these equalities is explained in 7.1.
Let $\mathbb{U}_n$ denote the following $n \times n$ matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{4.1.1}$$

The matrix $\mathbb{U}_n$ is obtained from the root system $\mathbb{A}_{2n}$ modulo involution of the respective Dynkin diagram. Here $\mathbb{U}$ stands for unreasonable (or un-necessary).

4.2. **The structure of** Mov **and** Ess**.** The following is a combination of the Wierzba's result [Wie03, 1.3] with 3.4.

Theorem 4.1. *Let $\pi : X \to Y$ be a local symplectic contraction (arbitrary dimension). Suppose that $N^1(X/Y)$ is generated by the classes of codimension 1 components E_α of the exceptional set of π, that is we are in situation of 3.5. Let e_α denote the numerical equivalence class of an irreducible component of a general fiber of $\pi_{|E_\alpha}$. Then the following holds:*

- *The classes of E_α are linearly independent so they form a basis of N_1.*
- *The opposite of the intersection matrix $-(e_\alpha \cdot E_\beta)$ is a direct sum of Cartan matrices of type associated to simple algebraic Lie groups (or algebras), and possibly, matrices of type $\mathbb{U}_n$.*
- *If moreover X is MDS over Y then* Mov(X/Y) *is dual, in terms of the intersection of $N^1(X/Y)$ and $N_1(X/Y)$, to the cone spanned by the classes of e_α. In particular* Mov(X/Y) *is simplicial.*

In short, the above theorem says that, apart of the case $\mathbb{U}_n$, that we do not expect to occur, the situation of arbitrary local symplectic contraction on the level of divisors and 1-cycles is very much like in the case of the contraction to nilpotent cone, which is the case of 7.4 and 7.5.

Conjecture 4.2. The case $\mathbb{U}_n$ should not occur. That is, there is no symplectic contraction $X \to Y$ with a codimension 2 locus of $\mathbb{A}_{2n}$ singularities of Y and a non-trivial numerical equivalence in X of curves in a general fiber of π over this locus.

We note that, since in dimension four X is an MDS over Y, in order to prove this conjecture it is enough to deal with the case when $X \to Y$ is elementary and in codimension 2 it is a contraction to $\mathbb{A}_2$ singularities. Indeed, we take an irreducible curve C_1 whose intersection with the irreducible divisor E_1 is (-1), which is the upper-right-hand corner of the matrix $\mathbb{U}_n$, see 4.1.1. The class of C_1 spans a ray on $\mathrm{Ess}(X/Y)$ and its dual $C_1^\perp \cap \mathrm{Mov}(X/Y)$ is a facet of $\mathrm{Mov}(X/Y)$. Hence we can choose an SQM model X' with a facet of $\mathrm{Nef}(X'/Y)$ contained in $C_1^\perp$. Thus there exists and elementary contraction of X' which contracts C_1 with exceptional locus which is (the strict transform of) E_1.

Corollary 4.3. *Suppose that the conjecture 4.2 is true. Then, for every local symplectic contraction $\pi : X \to Y$ satisfying the conditions of 3.5, there exists a semisimple Lie group and an identification of $N^1(X/Y)$ and $N_1(X/Y)$ with the real part of its Cartan algebra such that: (1) the intersection of the 1-cycles with classes of divisors is equal to the Killing form product, (2) the classes of irreducible essential curves spanning rays of* $\mathrm{Ess}(X/Y)$ *is identified with its primitive roots and (3) the cone* $\mathrm{Mov}(X/Y)$ *is identified with the Weyl chamber.*

Conjecture 4.4. Under the above identification the classes of (integral) 1-cycles should form the lattive Λ_R of roots, while the classes of divisors should make the lattice Λ_W of weights.

4.3. **Flopping classes, division of** Mov.

Theorem 4.5. *The subdivision of* $\mathrm{Mov}(X/Y)$ *into the nef subcones of different SQM models is obtained by cutting* $\mathrm{Mov}(X/Y)$ *with hyperplanes. That is, the union of the interiors of nef cones of all SQM models of X is equal to* $\mathrm{Mov}(X/Y) \setminus \bigcup \lambda_i^\perp$, *where $\{\lambda_i\}$ is a finite set of classes in $N_1(X/Y)$.*

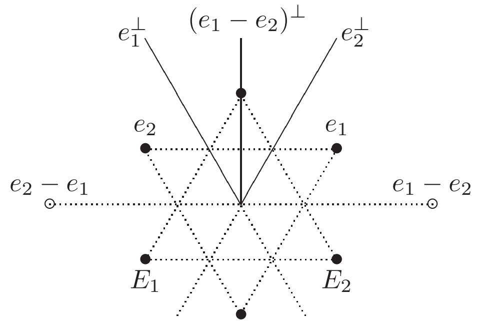
The λ_i's in the above theorem are determined up to multiplicity and they will be called *flopping classes*.

Proof. Take a ray R in the interior of the cone $\mathrm{Mov}(X/Y)$ which is an extremal ray in the nef cone of some model, say X. The exceptional locus of the contraction of X associate to R consists of a number of disjoint copies of $\mathbb{P}^2$, see the argument in the proof of (3.2) of [WW03]. Let W and W' be two walls of the subdivision of $\mathrm{Mov}(X/Y)$ into nef chambers of its SQM models, both containing R. The loci of curves determining W and W' are disjoint. Thus the flop with respect to the wall W does not affect the curves detemining W'. This implies that, as dividing walls, W' as well as W extend to hyperplanes containing R. □

4.4. **Examples of root systems.** Let us discuss examples of symplecitic contractions with 2-dimensional fibers. The resolution of $\mathbb{C}^4/\sigma_3$ (discussed in sections 6.3 and 7.2) is related to the root system $\mathbb{A}_1$ and it is not interesting. Similarly, the resolution of $\mathbb{C}^4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is related to $\mathbb{A}_1 \oplus \mathbb{A}_1$ and it has no flopping classes.

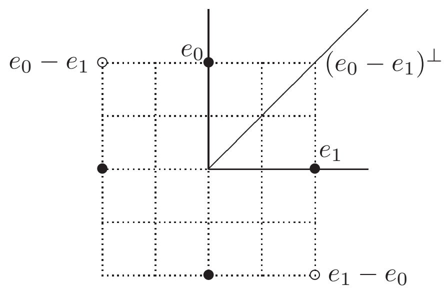
Example 4.6. The Lehn-Sorger resolution of $\mathbb{C}^4/BT$, see [LS08], is related to the root system $\mathbb{A}_2$ with generators e_1 and e_2. Then $v = \pm(e_1 - e_2)$ is the flopping system. Here is the picture of the weight lattice together with roots denoted by • and flopping classes denoted by ∘. The Mov cone (or Weyl chamber) is divided into two parts by the line orthogonal to the flopping class.

(4.4.2)



Example 4.7. Take the quotient $\mathbb{C}^4/\mathbb{Z}_2^2 \rtimes \mathbb{Z}_2$ and its Hilb-Chow resolution $X \to Y$. It is related to the decomposable root system $\mathbb{A}_1 \oplus \mathbb{A}_1$ with roots denoted by e_0 and e_1, respectively. The following picture describes a section of Mov together with its decomposition by flopping classes.

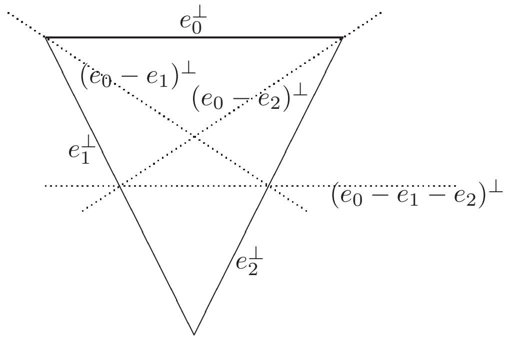
(4.4.3)



Example 4.8. Take the quotient $Y = \mathbb{C}^4/\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$ and its Hilb-Chow $X \to Y$ resolution. It is related to the decomposable root system $\mathbb{A}_2 \oplus \mathbb{A}_1$ with roots denoted by e_1, e_2 and e_0, respectively. The following picture describes a plane section of a 3-dimensional cone $\mathrm{Mov}(X/Y)$ (denoted by solid line segments) together with its decomposition by flopping classes (denoted by dotted line segments). The upper chamber in this picture is the nef cone $\mathrm{Nef}(X/Y)$. This situation will be discussed

in detail in 6.5 and 6.6.

(4.4.4)



5. Rational curves and differential forms

5.1. **The set-up.** Let $\pi : X \to Y$ be a local symplectic contraction of a 4-fold. We assume that we are in the situation of 3.5. In particular, the exceptional locus of π is a divisor D. This divisor, as well as its image surface, $S := \pi(D) \subset Y$, can be reducible. As above $0 \in S \subset Y$ denotes the unique point over which π can have a two dimensional fiber.
Our starting point is the paper of Wierzba [Wie03] (as well as the appendix of [SCW04]) to which we will refer. In particular the theorem 1.3 of [Wie03] says that the general non trivial fiber of π is a configuration of $\mathbb{P}^1$ with dual graph a Dynkin diagram. The components of these fibers are called essential curves in the previous section.
Choose an irreducible component of S, call it S'. Take an irreducible curve $C \simeq \mathbb{P}^1$ in a (general) fiber over a point in $S' \setminus \{0\}$ and let D' be the irreducible component of D which contains C; note that $\pi(D') = S'$ and S' may be (and usually is) non-normal. Let $\mathcal{V}' \subset Chow(X/Y)$ be an irreducible component of the Chow scheme of X containing C. By $\mathcal{V}$ we denote its normalization and $p : \mathcal{U} \to \mathcal{V}$ is the normalized pullback of the universal family over $\mathcal{V}'$. Finally, let $q : \mathcal{U} \to D' \subset X$ be the evaluation map, see e.g. [Kol96, I.3] for the construction. The contraction π determines a morphism $\tilde{\pi} : \mathcal{V} \to S'$, which is surjective because C was chosen in a general fiber over S'. We let $\mu : \mathcal{V} \to \widetilde{S}' \to S'$ be its Stein factorization. In particular $\widetilde{S}'$ is normal and $\nu : \widetilde{S}' \to S'$ is a finite morphism, ètale outside $\nu^{-1}(0)$, whose fibers are related to the orbits of the action of the group of automorphism of the Dynkin diagram, [Wie03, 1.3]. We will assume that μ is not an isomorphism which is equivalent to say that D' has a 2-dimensional fiber over 0. Also, since we are interested in understanding the local description of the contraction in analytic category we will assume that S' is analytically irreducible at 0 or that $\nu^{-1}(0)$ consists of single point. The exceptional locus of μ is $\mu^{-1}(\nu^{-1}(0)) = \bigcup_i V_i$ where $V_i \subset \mathcal{V}$ are irreducible curves.

(5.1.5)
$$\begin{array}{ccccc} \mathcal{U} & & \xrightarrow{\quad q \quad} & & D' \subset X \\ \downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle \pi} \\ \mathcal{V} & \xrightarrow{\mu} & \widetilde{S}' & \xrightarrow{\nu} & S' \subset Y \end{array}$$

If necessary, we can take $\mathcal{V}$ to be smooth, eventually by replacing it with its desingularization and $\mathcal{U}$ with the normalized fiber product. A general fiber of $p : \mathcal{U} \to \mathcal{V}$ is $\mathbb{P}^1$ while other fibers are, possibly, trees of rational curves. If C is an extremal curve, which by 3.4 and 3.2 is true for some SQM model of X, then $-D$ is ample on the extremal ray spanned by C and since $-D \cdot C \leq 2$ it follows that $p : \mathcal{U} \to \mathcal{V}$ is a $\mathbb{P}^1$ or conic bundle. Since any two SQM models of X are obtained by a sequence of Mukai flops, it follows that in a general situation $p : \mathcal{U} \to \mathcal{V}$ is obtained by a sequence of blows and blow-downs of a $\mathbb{P}^1$ or conic bundle.
In [Wie03] and [SCW04] it was proved that $S' \setminus \{0\}$ is smooth and that, on $\mathcal{V} \setminus \{(\nu \circ \mu)^{-1}(0)\}$, p is a $\mathbb{P}^1$-bundle. It was also showed, by pulling back the symplectic form via q and pushing it further down via p, that one can obtain a symplectic form on $S' \setminus \{0\}$. We will repeat their procedure in this more general case.

5.2. **The differentials.** Let us consider the derivative map $Dq : q^*\Omega_X \to \Omega_{\mathcal{U}}$. Its cokernel is a torsion sheaf, call it $\mathcal{Q}_{\Delta_2}$, supported on the set Δ_2, which is the set of points where q is not of maximal rank: by the purity theorem Δ_2 is a divisor. As for the kernel, let I be the ideal of D' in X and consider the sequence $q^*(I/I^2) \to q^*\Omega_X \to \Omega_{\mathcal{U}}$. The saturation of the image of the first map will be the kernel of the second map and it will be a reflexive sheaf of the form $\mathcal{O}_{\mathcal{U}}(-D' + \Delta_1)$, with Δ_1 being an effective divisor. In the above notation we can write the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{U}}(q^*(-D') + \Delta_1) \longrightarrow q^*\Omega_X \longrightarrow \Omega_{\mathcal{U}} \longrightarrow \mathcal{Q}_{\Delta_2} \longrightarrow 0. \tag{5.2.6}$$

We have another derivation map into $\Omega_{\mathcal{U}}$, namely $Dp : p^*\Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}}$. It fits in the exact sequence

$$p^*\Omega_{\mathcal{V}} \longrightarrow \Omega_{\mathcal{U}} \longrightarrow \Omega_{\mathcal{U}/\mathcal{V}} \longrightarrow 0, \tag{5.2.7}$$

whose dual sequence is

$$0 \longrightarrow T_{\mathcal{U}/\mathcal{V}} \longrightarrow T_{\mathcal{U}} \longrightarrow p^*T_{\mathcal{V}} \tag{5.2.8}$$

The symplectic form on X, that is ω_X, gives an isomorphism $\omega_X : T_X \to \Omega_X$. We consider the following diagram involving morphism of sheaves over $\mathcal{U}$ appearing in the above sequences.

$$\begin{array}{ccccccccc} T_{\mathcal{U}/\mathcal{V}} & \longrightarrow & T_{\mathcal{U}} & \xrightarrow{(Dp)^*} & p^*(T_{\mathcal{V}}) & \overset{p^*(\omega_{\mathcal{V}})}{\dashrightarrow} & p^*(\Omega_{\mathcal{V}}) & & \\ & & \downarrow{\scriptstyle (Dq)^*} & & & & \downarrow{\scriptstyle Dp} & & \\ & & q^*T_X & \xrightarrow[q^*(\omega_X)]{} & q^*\Omega_X & \xrightarrow{Dq} & \Omega_{\mathcal{U}} & \longrightarrow & \Omega_{\mathcal{U}/\mathcal{V}} \end{array} \tag{5.2.9}$$

We claim that the dotted arrow exists and it is obtained by a pull back of a two form $\omega_{\mathcal{V}}$ on $\mathcal{V}$, and it is an isomorphism outside the exceptional set of μ which is $\bigcup_i V_i$. Indeed, the composition of arrows in the diagram which yields $T_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ is given by the 2-form $Dq(\omega_X)$ and it is zero on $T_{\mathcal{U}/\mathcal{V}}$ because this is a torsion free sheaf and its restriction to any fiber of p outside $\bigcup_i V_i$ (any fiber of p is there a

$\mathbb{P}^1$) is $\mathcal{O}(2)$ while the restriction of $\Omega_{\mathcal{U}}$ is $\mathcal{O}(-2)\oplus\mathcal{O}\oplus\mathcal{O}$. By the same reason the composition $T_{\mathcal{U}}\to\Omega_{\mathcal{U}}\to\Omega_{\mathcal{U}/\mathcal{V}}$ is zero since $T_{\mathcal{U}}$ on any fiber of p outside $\bigcup_i V_i$ is $\mathcal{O}(2)\oplus\mathcal{O}\oplus\mathcal{O}$ while $\Omega_{\mathcal{U}/\mathcal{V}}$ is $\mathcal{O}(-2)$. Thus the map $Dq(\omega_X): T_{\mathcal{U}}\to\Omega_{\mathcal{U}}$ factors through $p^*(T_{\mathcal{V}})\to p^*(\Omega_{\mathcal{V}})$ and, as a result, $Dq(\omega_X)=Dp(\omega_{\mathcal{V}})$, for some 2-form $\omega_{\mathcal{V}}$ on $\mathcal{V}$. Since Dq is of maximal rank outside of $p^{-1}(\bigcup_i V_i)$ and p is there just a $\mathbb{P}^1$-bundle it follows that $\omega_{\mathcal{V}}$ does not assume zero outside the exceptional set of μ. Hence $K_{\mathcal{V}}=\sum a_iV_i$ with $a_i\geq 0$ being the discrepancy of V_i. Note that the above argument is essentially from [SCW04, Sect. 4.1] or [Wie03, Sect. 5].

Theorem 5.1. *The surface $\widetilde{S}'$ has at most Du Val (or $\mathbb{A}-\mathbb{D}-\mathbb{E}$) singularity at $\nu^{-1}(0)$ and $\mu:\mathcal{V}\to\widetilde{S}'$ is its resolution, possibly non-minimal. In particular every V_i is a rational curve. If a component V_i has positive discrepancy or, equivalently, the form $\omega_{\mathcal{V}}$ vanishes along V_i, then $p^{-1}(V_i)\subset\Delta_2$.*

Proof. The first statement follows from the discussion preceeding the proposition. To get the next one, note that over $\mathcal{U}$ we have $Dq(\omega_X)=Dp(\omega_{\mathcal{V}})$ and $\omega_{\mathcal{V}}$ is zero at any component of $\bigcup_i V_i$ of positive discrepancy. Since ω_X is nondegenerate this equality implies that Dq is of rank ≤ 2 on the respective component of $p^{-1}(\bigcup V_i)$. □

We note that although the surface $\widetilde{S}'$ is the same for all the symplectic resolutions of Y, the parametric scheme for lines, which is a resolution of $\widetilde{S}'$ may be different for different SQM models, see 6.5 for an explicit example.

Proposition 5.2. *Suppose that the map p is of maximal rank in codimension 1. Then the p-inverse image of the set of positive discrepancy components of $\bigcup_i V_i$ coincides with the set where the rank of q drops. That is, Δ_2 is the pullback of the zero set of $\omega_{\mathcal{V}}$.*

Proof. We have the following injective morphism of sheaves $q^*(\omega_X)\circ(Dq)^*(T_{\mathcal{U}/\mathcal{V}})\hookrightarrow\mathcal{O}_{\mathcal{U}}(-p^*D+\Delta_1)\hookrightarrow q^*\Omega_X$ which follows, as already noted, because of the splitting type of $\Omega_{\mathcal{U}}$. We claim that this implies the isomorphism of line bundles $T_{\mathcal{U}/\mathcal{V}}\simeq\mathcal{O}_{\mathcal{U}}(-p^*D+\Delta_1)$. Indeed, the evaluation map of the universal family over the Chow scheme is isomorphic on the fibers, hence $(Dq)^*$ is of maximal rank along $T_{\mathcal{U}/\mathcal{V}}$ is codimension 1 at least, hence the desired isomorphism.
Now, since p is submersive in codimension 1, because of the sequence 5.2.7 we can write $det\Omega_{\mathcal{U}}=p^*(K_{\mathcal{V}})\otimes\Omega_{\mathcal{U}/\mathcal{V}}$ and consequently, because of the sequence 5.2.6, we get

$$c_1(\mathcal{Q}_{\Delta_2})=c_1(\mathcal{O}_{\mathcal{U}}(-p^*D+\Delta_1))-c_1(T_{\mathcal{U}/\mathcal{V}})+c_1(p^*(K_{\mathcal{V}}))=c_1(p^*K_{\mathcal{V}})=p^*(\sum a_i[V_i])$$

□

5.3. **Vertical slicing.** The first of the following two results is essentially known, c.f. [Kal06, 2.3] and also [Wie03, 1.2(ii), 1.4.]. We restate it and reprove it in the form suitable for the subsequent corollary.

Proposition 5.3. *Suppose that $\pi: X\to Y$ is a symplectic contraction with $\dim X=2n$. Let $Z\subset X$, with $\operatorname{codim} Z=m$, be a (irreducible) maximal cycle with $S=\pi(Z)$, $\operatorname{codim} S=2m$. The fibers of $\pi_{|Z}: Z\to S$ are isotropic (with respect to ω_X) and, moreover, over an open and dense set $S_0\subset S$ there exists a symplectic form ω_S such that over $\pi_{|Z}^{-1}(S_0)$ we have $D\pi(\omega_S)=\omega_{X|Z}$*

Proof. The proof that ω_X restricted to fibers of π is zero so that they are isotropic (or lagrangian) is in [WW03, 2.20]. Let $\iota : Z \to X$ be the embedding. Then we have the following version of diagram 5.2.9

$$\begin{array}{ccccccc} T_{Z/S} & \longrightarrow & T_Z & \xrightarrow{(D\pi)^*} & \pi^*(T_S) & \overset{\pi^*(\omega_S)}{\cdots\cdots>} & \pi^*(\Omega_S) & & \\ & & \Big\downarrow{\scriptstyle (D\iota)^*} & & & & \Big\downarrow{\scriptstyle D\pi} & & \\ & & \iota^*T_X & \xrightarrow[\iota^*(\omega_X)]{} & \iota^*\Omega_X & \xrightarrow{D\iota} & \Omega_Z & \longrightarrow & \Omega_{Z/S} \end{array} \tag{5.3.10}$$

We claim the existence of ω_S. The composition $T_{Z/S} \to \Omega_{Z/S}$ is trivial since fibers of π are isotropic. On the other hand the induced maps $\pi^*(T_S) \to \Omega_{Z/S}$ and $T_{Z/S} \to \pi^*\Omega_S$ are zero: indeed, otherwise we would have nonzero 1 forms on a generic fiber of $\pi_{|Z}$, which would contribute to the first cohomology of the fiber (via the Hodge theory on the simplicial resolution of the fiber) which contradicts [Kal06, 2.12].
Thus the dotted arrow in the above diagram is well defined and it satisfies $D\pi(\omega_S) = D\iota(\omega_X)$ for a two form ω_S defined over a smooth subset S_0 of S. Moreover the form ω_S is of maximal rank for the dimensional reasons. □

The following corollary is a symplectic version of [AW98, 1.3].

Corollary 5.4. [Vertical slicing] *In the situation of 5.3 let $H_1, \dots H_{2n-2m}$ be general irreducible divisors in Y meeting in a general point $s \in S$. Letting $Y' = H_1 \cap \dots \cap H_{2n-2m}$ and $X' = \pi^{-1}(Y')$ and possibly shrinking Y' and X' for a neighbourhood of s we get $\pi' = \pi_{|X'} : X' \to Y'$ a local symplectic contraction of $2m$-fold with an exceptional fiber $\pi^{-1}(s)$ of dimension m.*

Proof. Since π is crepant it is enough to show that the restriction of ω_X to X' is nondegenerate at a point over s in order to claim that it is symplectic over the whole X' (after possibly shrinking Y' to a neighbourhood of s). To this end we consider the following commuting diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{X'|Z_s} & \longrightarrow & T_{X|Z_s} & \longrightarrow & (N_{X'/X})_{Z_s} = (T_S)_s \otimes \mathcal{O}_{Z_s} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \omega_{X|X'}} & & \Big\downarrow{\scriptstyle \omega_X} & & \Big\downarrow{\scriptstyle D\pi(\omega_S)} & & \\ 0 & \longleftarrow & \Omega_{X'|Z_s} & \longleftarrow & \Omega_{X|Z_s} & \longleftarrow & (N^*_{X'/X})_{|Z_s} = (T^*_S)_s \otimes \mathcal{O}_{Z_s} & \longleftarrow & 0 \end{array} \tag{5.3.11}$$

Here $Z_s = Z \cap X'$ is a complete intersection, hence$(N_{X'/X})_{|Z_s} = N_{Z_s/Z}$ which yields the identifications in the last non-zero column of the diagram. The right-hand-side vertical arrow follows because of 5.3 where we have also shown that is an isomorphism. This implies that the left-hand-side vertical arrow is an isomorphism too. □

6. Quotient symplectic singularities, examples

6.1. **Preliminaries.** In this section $G < Sp(\mathbb{C}^4) =: Sp(4)$ is a finite subgroup preserving a symplectic form. We will discuss some examples in which $Y := \mathbb{C}^4/G$ admits a symplectic resolution $\pi : X \to Y$. We have the following two fundamental results about such resolutions, the latter one known as McKay correspondence.

Theorem 6.1. (c.f. [Ver00]) *If Y admits a symplectic resolution then G is generated by symplectic reflections, that is elements whose fixed points set is of codimension 2.*

Theorem 6.2. (c.f. [Kal02]) *The homology classes of the maximal cycles (as defined in 2.2) form a basis of rational homology of X and they are in bijection with conjugacy classes of elements of G.*

On the other hand we have the following immediate observation (for further details see for instance section 3.2 in [AW10]).

Lemma 6.3. *Let $S' \subset Y$ be a component of the codimension 2 singular locus associated to the isotropy group $H < G$. Then H is one of the $\mathbb{A}-\mathbb{D}-\mathbb{E}$ groups (a finite subgroup of $SL(\mathbb{C}^2)$) consisting of symplectic reflections. The normalization of S' has a quotient singularity by the action of $W(H) = N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G.*

6.2. **Direct product resolution.** Let $H_1, H_2 < SL(2)$ be finite subgroups and consider $G := H_1 \times H_2$ acting on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. Let $\pi_i : S_i \to \mathbb{C}^2/H_i$ be minimal resolutions and $n_i = |H_i| - 1$ the number of exceptional rational curves in S_i. The product morphism $\pi = \pi_1 \times \pi_2 : X := S_1 \times S_2 \to Y := \mathbb{C}^4/G$ is a symplectic resolution with the central fiber isomorphic to the product of the exceptional loci of π_i. In particular X does not admit any flop and $\mathrm{Mov}(X/Y) = \mathrm{Nef}(X/Y)$. Every component of $Chow(X/Y)$ containing an exceptional curve of π_i is isomorphic to S_j, with $i \neq j \in \{1, 2\}$.

6.3. **Elementary contraction to $\mathbb{C}^4/\sigma_3$.** A symplectic resolution of the quotient $\mathbb{C}^4/\sigma_3$, where σ_3 is a group of permutation of 3 elements, can be obtained as a section of the Hilbert-Chow morphism $\tau : Hilb^3(\mathbb{C}^2) \to (\mathbb{C}^2)^{(3)}$ in 2.3. This is a local version of Beauville's construction, [Bea83], and a special case of 5.4. There are three conjugacy classes in σ_3 which are related to three maximal cycles, of complex dimension 4, 3 and 2, each related to a 1-dimensional group of homology for the resolution $\pi : X \to Y = \mathbb{C}^4/\sigma_3$.
Since the normalizer of the order 2 element in σ_3 (the reflection, if one thinks about σ_3 as the dihedral group) is trivial, by lemma 6.3 it follows that the normalization of the singular locus S of Y is smooth. Hence, by 5.2 we can compute both the parametrizing scheme for rational curves in X and the respective universal family. That is, the parametrizing scheme $\mathcal{V}$ is just a blow-up of the normalization of S, the evaluation map $q : \mathcal{U} \to X$ drops its rank over 0 and the exceptional divisor of π, which is the image of q is non-normal over 0.
More explicit calculations are done in section 7.2.

6.4. **Wreath product.** Let $H < SL(2)$ be a finite subgroup and let $G := H^{\times 2} \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ interchanges the factors in the product. We write $G = H \wr \mathbb{Z}_2$. Note that $\mathbb{Z}_{n+1} \wr \mathbb{Z}_2$ has another nice presentation, namely $(\mathbb{Z}_{n+1})^{\times 2} \rtimes \mathbb{Z}_2 = D_{2n} \rtimes \mathbb{Z}_n$, where $\mathbb{Z}_n$ acts on the the dihedral group D_{2n} of the regular n-gon by rotations.
We consider the projective symplectic resolution described in 2.3 (with $n = 2$):

$$\pi : X := Hilb^2(S) \to S^{(2)} \to (\mathbb{C}^2/H)^{(2)} := Y$$

where $\nu : S \to \mathbb{C}^2/H$ is the minimal resolution with the exceptional set $\bigcup_i C_i$, where C_i, $i = 1, ..., k$, are (-2)-curves.

The morphism $\tau : Hilb^2(S) \to S^{(2)}$ is just a blow-up of $\mathbb{A}_1$ singularities (the image of the diagonal under $S^2 \to S^{(2)}$) with irreducible exceptional divisor E_0 which is a $\mathbb{P}^1$ bundle over S. We set $S' = \pi(E_0)$. By E_i, with $i = 1, \dots, k$ we denote the strict transform, via τ, of the image of $C_i \times S$ under the map $S^2 \to S^{(2)}$. By e_i we denote the class of an irreducible component of a general fiber of $\pi_{|E_i}$. The image $\pi(E_i)$ for $i \geq 1$ is the surface $S'' \simeq C^2/H$. The singular locus of Y is the union $S = S' \cup S''$.
The irreducible components of $\pi^{-1}(0)$ are described in the following.

- $P_{i,i}$, for $i = 1, ..., k$. They are the strict transform of $C_i^{(2)}$ via τ. They are isomorphic to $\mathbb{P}^2$.
- $P_{i,j}$, for $i, j = 1, ..., k$ and $i < j$. They are the strict transform via τ of the image of $C_i \times C_j$ under the morphism $S^2 \to S^{(2)}$. They are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ if $C_i \cup C_j = \emptyset$ and to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ if $C_i \cup C_j = \{x_i\}$.
- Q_i, for $i = 1, ..., k$. They are the preimage $\tau^{-1}\Delta_{C_i}$, where Δ_{C_i} is the diagonal embedding of C_i in $S^{(2)}$. It is isomorphic to $\mathbb{P}(T_{S|C_i}) = \mathbb{P}(\mathcal{O}_{C_i}(2) \oplus \mathcal{O}_{C_i}(-2))$, a Hirzebruch surface F_4.

Let us also describe some special intersections between these components. Namely, $P_{i,i}$ intersects Q_i along a curve which is a (-4)-curve in Q_i and a conic in $P_{i,i}$. If $C_i \cap C_j = \{x_i\}$ then $P_{i,j}$ intersect $P_{i,i}$ (respectively $P_{j,j}$) along a curve which is a (-1) curve in $P_{i,j}$ and a line in $P_{i,i}$ (respectively in $P_{j,j}$). Moreover in this case $P_{i,j}$ intersect Q_i (respectively Q_j) in a curve which is a (-1) curve in $P_{i,j}$ and a fiber in Q_i (respectively Q_j).
The next lemma is straightforward, a proof of it can be found in [Fu06b, Lemma 4.2].

Lemma 6.4. *The strict transform of Q_i under any sequence of Mukai flops along components in $\pi^{-1}(0)$ is not isomorphic to $\mathbb{P}^2$.*

6.5. **Resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$.** The Figure 1 presents a “realistic” description of configurations of components in the special fiber of symplectic resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$. By abuse, the strict transforms of the components and the results of the flopping of $\mathbb{P}^2$’s are denoted by the same letters.
The position of these configurations in Figure 1 is consistent with the decomposition of the cone $\mathrm{Mov}(X/Y)$ presented in the diagram 4.4.4. In particular, the configuration at the top is associated to the Hilb-Chow resolution. Note that the central configuration of this diagram cointains three copies of $\mathbb{P}^2$, denoted P_{ij}, which contain lines whose classes are $e_0 - e_1$, $e_0 - e_2$ and $e_1 + e_2 - e_0$.
On the other hand, the configuration in the bottom is associated to the resolution which can be factored by two different divisorial elementary contractions of classes e_1 and e_2. In fact, contracting both e_1 and e_2 is a resolution of $\mathbb{A}_2$ singularities which is a part of a resolution of Y which comes from presenting $\mathbb{Z}_3 \wr \mathbb{Z}_2 = (\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$ as $D_6 \rtimes \mathbb{Z}_3$. That is, X is then obtained by first resolving the singularities of the action of $D_6 = \sigma_3$ and then by resolving singularities of $\mathbb{Z}_3$ action on this resolution. We will call such X a $D_6 \rtimes \mathbb{Z}_3$-resolution.
This example is convenient for understanding the contents of the theorem 5.1 and of the proposition 5.2. We refer to diagram 5.1.5 and let S' and S'' be the closure of the locus of $\mathbb{A}_1$ and $\mathbb{A}_2$ singularities in $Y = \mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$. From lemma 6.3 we find out that the normalization of S' as well as S'' has a singularity of type $\mathbb{A}_2$.

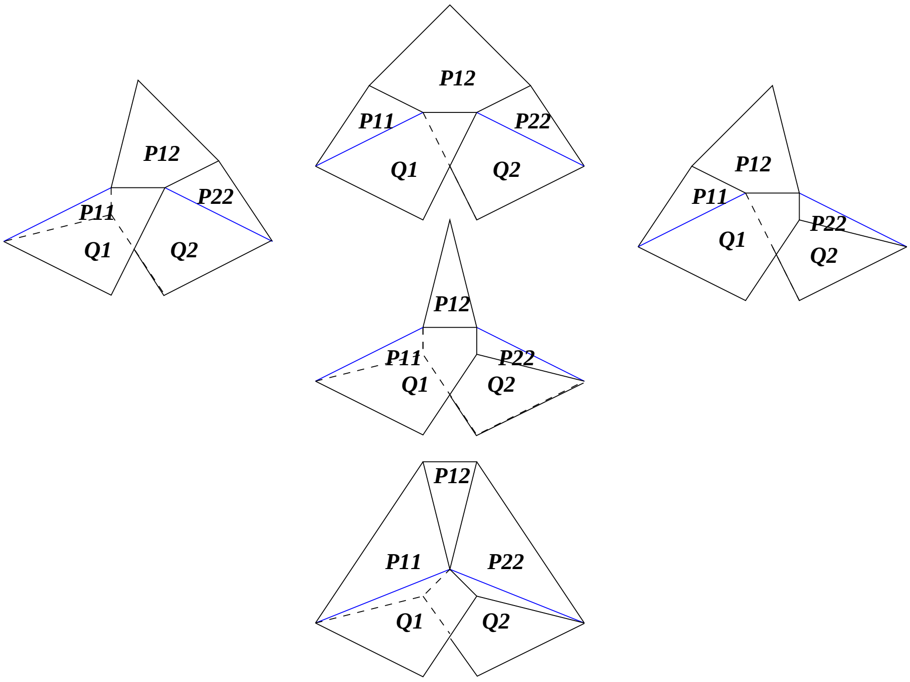


FIGURE 1. Components of the central fiber in resolutions of $\mathbb{C}^4/(\mathbb{Z}_3 \wr \mathbb{Z}_2)$

By $\mathcal{V}_0$ we denote the component of $Chow(X/Y)$ dominating S' and parametrizing curves equivalent to e_0 while by $\mathcal{V}_1$ and $\mathcal{V}_2$ we denote components dominating S'' parameterizing deformations of e_1 and e_2. The surfaces $\mathcal{V}_i$ may depend on the resolution and, in fact, while $\mathcal{V}_1$ and $\mathcal{V}_2$ remain unchanged, the component $\mathcal{V}_0$ will change under flops.

Lemma 6.5. *If X is the Hilb-Chow resolution then $\mathcal{V}_0$ is the minimal resolution of $\mathbb{A}_2$ singularity. If X is the $D_6 \rtimes \mathbb{Z}_3$-resolution then $\mathcal{V}_0$ is non-minimal, with one (-1) curve in the central position of three exceptional curves.*

Proof. The first statement is immediate. To see the second one, note that we have the map of $\mathcal{V}_0$ to Chow of lines in the resolution of $\mathbb{C}^4/\sigma_3$ divided by $\mathbb{Z}_3$ action. The $\mathbb{Z}_3$-action in question is just a lift up of the original linear action on the fixed point set of rotations in $\sigma_3 = D_6$ hence $\mathcal{V}_0$ resolves 2 cubic cone singularities associated to eigenvectors of the original action. □

One may verify that the positive discrepancy component of the exceptional set in the $\mathcal{V}_0$ in the $D_6 \rtimes \mathbb{Z}_3$-resolution parametrizes curves consisting of three components: $\mathbb{Q}_2 \cap P_{11}$, $Q_1 \cap P_{22}$ and a line in P_{12}, whose classes are, respectively, e_2, e_1 and $e_0 - (e_1 + e_2)$.

6.6. **Resolutions of $\mathbb{C}^4/(\mathbb{Z}_{n+1} \wr \mathbb{Z}_2)$.** Let us use the notation introduced in theorem 4.1, corollary 4.3 and in the set up of 6.4 for the case $H = \mathbb{Z}_{n+1}$. In particular, for $i = 1, \ldots, n$ the classes e_i are identified to simple roots associated to consequtive nodes of the Dynkin diagram $\mathbb{A}_n$.

Theorem 6.6. *Let* $X \to Y = \mathbb{C}^4/(\mathbb{Z}_{n+1} \wr \mathbb{Z}_2)$ *be a symplectic resolution as above. The division of* $\mathrm{Mov}(X/Y) = \langle e_0, \dots, e_n\rangle^\vee$ *into Mori chambers is defined by hyperplanes* $\lambda_{ij}^\perp$ *for* $1 \le i \le j \le n$*, where* $\lambda_{ij} = e_0 - (e_i + e_{i+1} + \cdots + e_{j-1} + e_j)$.

A proof of this theorem will occupy the rest of this section.
We know one Mori chamber of $\mathrm{Mov}(X/Y)$, the one associated to the Hilbert-Chow resolution. The faces of this chamber are supported by $e_0^\perp$ and by $-\lambda_{ii}^\perp = (e_i - e_0)^\perp$, see e.g. the above discussion. Thus, in particular, if $\lambda \in N_1(X)$ is a flopping class then $\lambda^\perp$ does not meet the relative interior of the face $\mathrm{Mov}(X) \cap e_0^\perp$.
On the other hand, $\mathrm{Mov}(X/Y) = \mathrm{Mov}(X/Y) \cap e_0^\perp + \mathbb{R}_{\ge 0} \cdot (-E_0)$. Thus, if we take any D_0 in the relative interior of $\mathrm{Mov}(X) \cap e_0^\perp$ then, by the above observation, the half-line $D_0 + \mathbb{R}_{\ge 0} \cdot (-E_0)$ must meet the hyperplane $\lambda^\perp$, for any flopping class λ. Hence the theorem will be proved if, for a choice of D_0, we will show that all hyperplanes $\lambda^\perp$ that $D_0 + \mathbb{R}_{\ge 0} \cdot (-E_0)$ meets actually come from the classes λ_{ij}.
Let us choose a sequence (a vector) of n positive numbers $\overline{\beta} = (\beta_i)$ such that $\beta_1 + \cdots + \beta_{i-1} < \beta_i$, for $i = 2, \dots, n$. We set $\gamma_{ij} = \beta_i + \cdots + \beta_j$. Then, by our assumption,

$$\gamma_{11} < \gamma_{22} < \gamma_{12} < \gamma_{33} < \gamma_{23} < \gamma_{13} < \gamma_{44} < \gamma_{34} \cdots \tag{6.6.12}$$

Let A be the intersection matrix for the root system $\mathbb{A}_n$. The matrix $-A$ is negative definite therefore there exists a unique vector $\overline{\alpha} = (\alpha_i)$ such that $(-A)\cdot\overline{\alpha} = \overline{\beta}$. If we now set $D_0 = \sum_i \alpha_i E_i$ then $D_0 \cdot e_0 = 0$ and $D_0 \cdot e_i = \beta_i > 0$ for $i = 1, \dots, n$ hence D_0 is in the relative interior of $\mathrm{Mov}(X) \cap e_0^\perp$. What is more, if we set $D_t = D_0 - (t/2)E_0$ then $D_t \cdot \lambda_{ij} = t - \gamma_{ij}$; so that γ_{ij} is the threshold value of t for the form λ_{ij} on the half-line $\{D_t : t \in \mathbb{R}_{\ge 0}\}$. The SQM model of X on which the divisor D_t is ample will be denoted by X_t.
Now our theorem is equivalent to saying that the models X_t are in bijection with connected components (open intervals) in $\mathbb{R}_{>0} \setminus \{\gamma_{ij}\}$. This can be verified by starting from X_{Hilb} associated to interval $(0, \gamma_{11})$ and proceeding inductively as it follows. Let t be in the interval $(\gamma_{ij}, \gamma_{i'j'})$, where γ_{ij} and $\gamma_{i'j'}$ are consecutive numbers in the sequence of γ's. We verify first that the $\mathbb{P}^2$-s which are in the exceptional locus of X_t have lines whose classes are only of type $\pm\lambda_{rs}$; secondly that pairs (i, j) and (i', j') are among those (r, s) which occur on X_t. The sign of $\pm\lambda_{rs}$ will depend on the position of γ_{rs} with respect to t. Hence we flop the $\mathbb{P}^2$ with lines of type $-\lambda_{i'j'}$ and proceed to the next interval. Note that with this single flop we keep the (relative) projectivity of the model (over Y). The argument will stop when X_t contains only one $\mathbb{P}^2$, with lines in the class $+\lambda_{1n}$.
We run this algorithm in the next section.

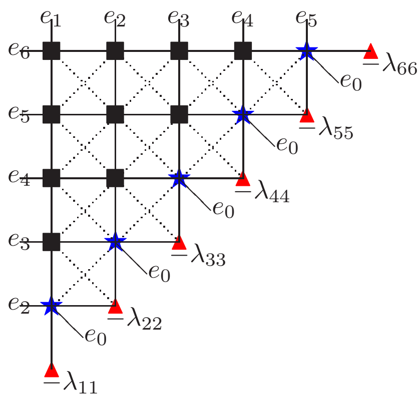
6.7. **Explicit flops.** The following are the diagrams of incidence of flopping components of the special fiber of a resolution of the 4-dimensional symplectic singularity coming from the action of $\mathbb{Z}_n \wr \mathbb{Z}_2$, where here $n \le 7$. We ignore the components which cannot become bP^2, i.e. the ones which are not flopped, which are isomorphic to F_4.
The curves are described by their class in cohomology. In particular, $e_0, e_1, \dots, e_n$ denote the classes of essential curves, where e_0 is the class of a fiber over the surface of $\mathbb{A}_1$ singularities and $e_1, \dots, e_n$ are components of the fiber over $\mathbb{A}_n$ singularity. Moreover for $0 \le i \le j \le n$ by λ_{ij} we denote $e_0 - (e_i + \cdots + e_j)$.

The incidence of components in terms of points is denoted by dotted line segments, while in terms of curves by solid line segments. The isomorphisms classes of surfaces are denoted by the following codes: ▲ $=\mathbb{P}^2$, ◆ $=F_1$, ■ $=\mathbb{P}^1\times\mathbb{P}^1$ and ★ denotes blow-up of $\mathbb{P}^2$ in two points (or $\mathbb{P}^1\times\mathbb{P}^1$ in one point). Finally, γ_{ij} are the threshold values associated to flops in 6.6.12 and $(\gamma_{ij},\gamma_{i'j'})$ denotes the isomorphism class of the resolution in the interval bounded by these thresholds.

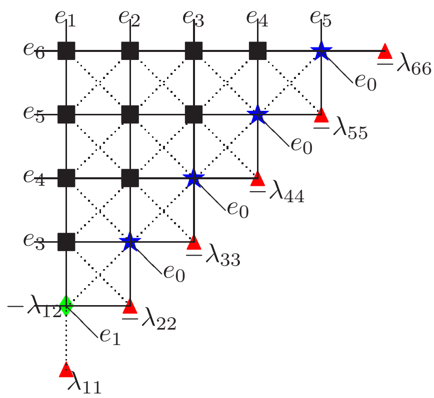
We note that, in the diagrams below, e_i's appear as classes of rulings of quadrics as well as F_1's, while $\pm\lambda_{ij}$'s are classes of lines in $\mathbb{P}^2$ or sections of F_1's. In fact, one can easily compute the classes of all edges (incidence curves) in our diagrams which we did not label for the sake of clarity of the picture. For example, incidence curves for $\mathbb{P}^1\times\mathbb{P}^1$, represented as edges of our diagrams at the vertex denoted by ■, have the same classes at the opposite ends of the vertex: e.g. the class of –■ is the same as of ■–. Also, if the class of –◆ is λ_{ij} and the class of ◆– is $\lambda_{i+1,j}$ then the class of ruling, e.g. the class ◆, is equal to $\lambda_{ij}-\lambda_{i+1,j}=e_{i+1}$.

Finally, let us not that similar diagrams are in [Fu06a]. The method used in that paper is similar to ours since the starting point is Hilb-Chow resolution but there each step involves several flops. However, the resulting diagrams in [Fu06a] are not quite correct since they imply that the components of the exceptional fiber in the final chamber are one $\mathbb{P}^2$ and all the rest F_1's.

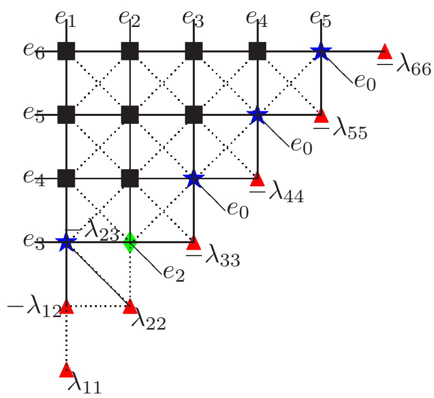
$(0,\gamma_{11})$



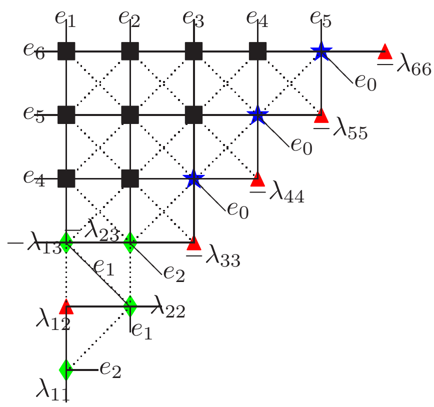
$(\gamma_{11},\gamma_{22})$



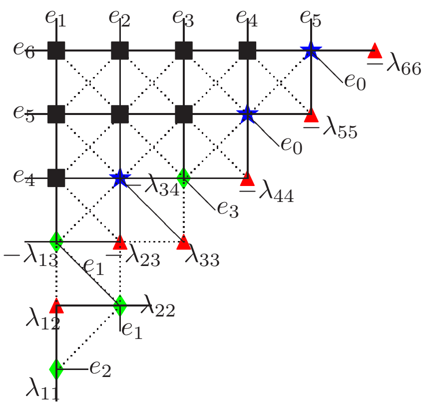
$(\gamma_{22},\gamma_{12})$



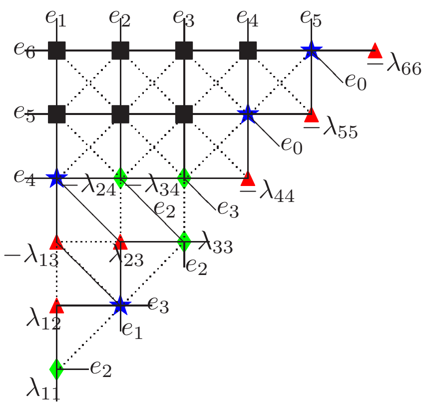
$(\gamma_{12},\gamma_{33})$



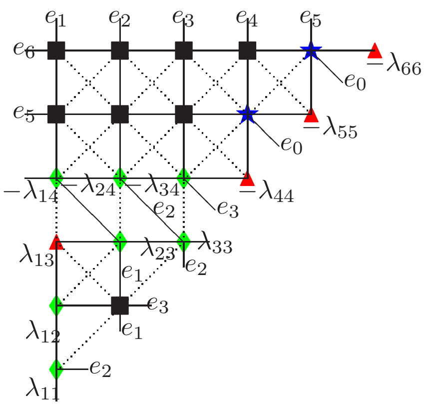
$(\gamma_{33}, \gamma_{23})$



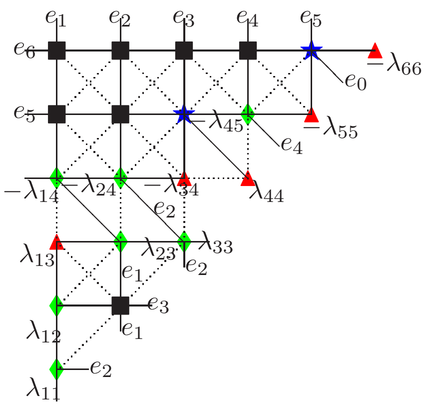
$(\gamma_{23}, \gamma_{13})$



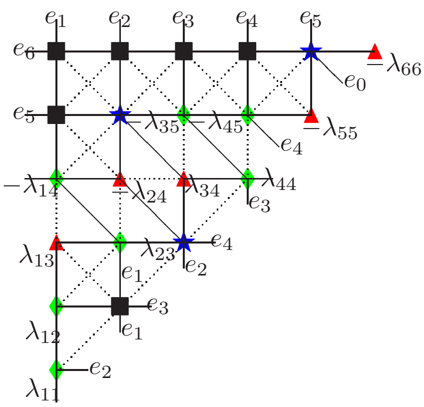
$(\gamma_{13}, \gamma_{44})$



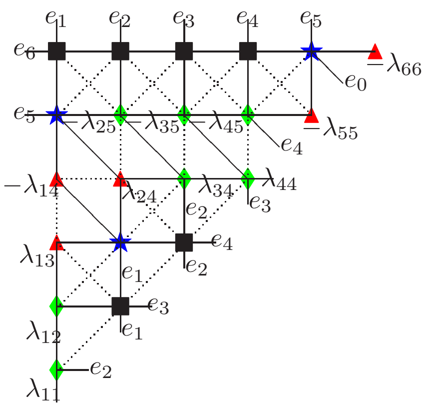
$(\gamma_{44}, \gamma_{34})$



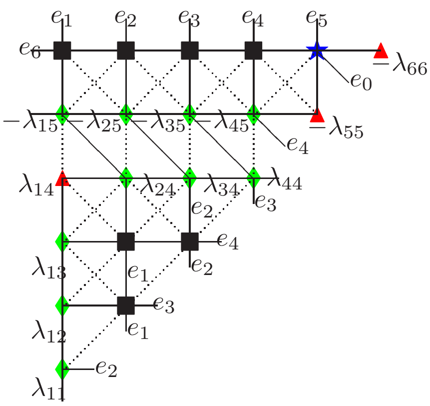
$(\gamma_{34}, \gamma_{24})$



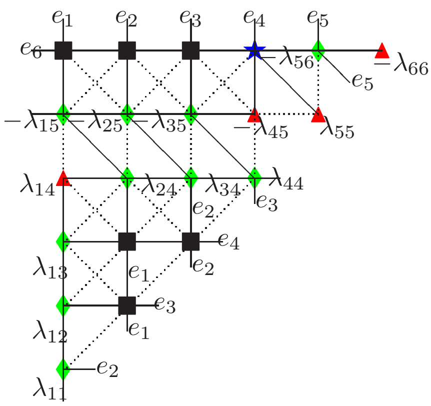
$(\gamma_{24}, \gamma_{14})$



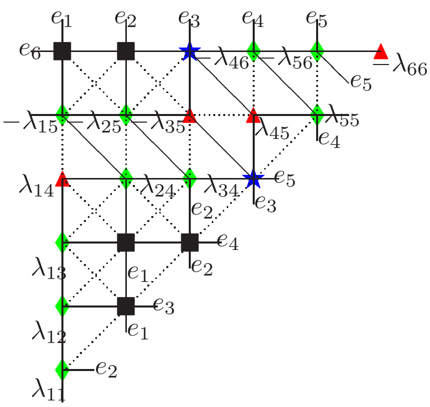
$(\gamma_{14}, \gamma_{55})$



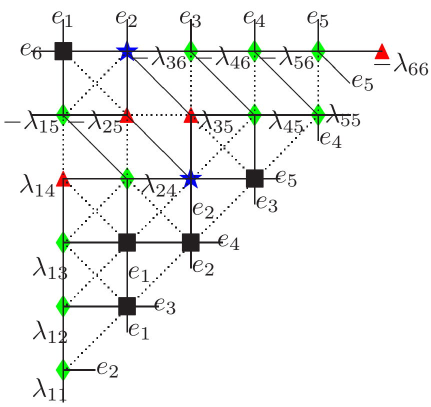
$(\gamma_{55}, \gamma_{45})$



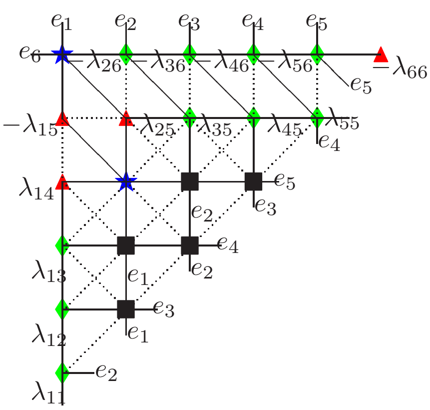
$(\gamma_{45}, \gamma_{35})$



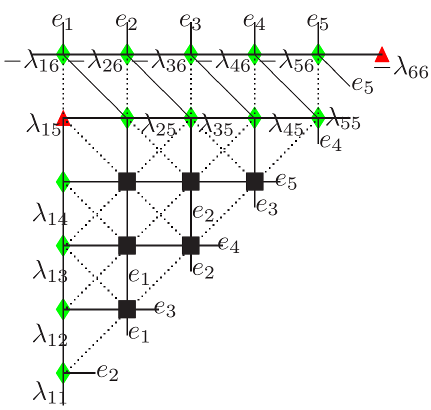
$(\gamma_{35}, \gamma_{25})$



$(\gamma_{25}, \gamma_{15})$

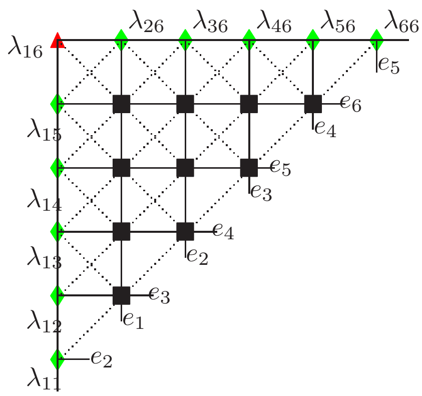


$(\gamma_{15}, \gamma_{66})$



$(\gamma_{66}, \gamma_{16})$ $(\gamma_{16}, +\infty)$

6 more flops in the upper row



7. Appendix

7.1. **Contraction to the nilpotent cone.** In this subsection we recall known facts about flag varieties of simple Lie groups and contractions to the nilpotent cone. This subject is classical and well documented, see e.g. [Slo80] or [CM93] and references therein. However, our point of view is somehow more geometric, related to homogeneous varieties, in the spirit of [Ott95], and directed on understanding the picture at the level of the related root systems. We refer to [TY05, Ch. 18] for generalities on root systems.

Let G be a complex simple algebraic group with the Lie algebra $\mathfrak{g}$. By R we denote the set of roots of $\mathfrak{g}$ and consider the lattices of roots and of weights $\Lambda_R \subset \Lambda_W$ of the algebra (or group) in question and we let $V = \Lambda_R \otimes \mathbb{R}$. By B we denote a Borel subgroup of G and $F = G/B$ is its flag variety. It is known that we have a natural isomorphism $\operatorname{Pic} F \simeq \Lambda_W$ under which $\operatorname{Nef}(F) \subset N^1(F)$ is identified with the Weyl chamber in V. Under this identification any irreducible representation U_w of G with the highest weight w is the complete linear system on F of a nef line bundle and the associated map $F \to \mathbb{P}(U_w)$ maps F to the unique closed orbit.

Moreover, the sum of the positive roots $\rho = \sum_{\alpha \in R^+} \alpha$ can be identified with the anticanonical class $-K_F$ and the Weyl formula, describing the dimension of irreducible representations, yields the Hilbert polynomial on $\operatorname{Pic} F$. That is, for every $\lambda \in \Lambda_W$ the dimension formula, or the Euler characteristic of the respective line bundle on F, can be written as a polynomial

$$H(\lambda) = \prod_{\alpha \in R^+} \frac{((\lambda + \rho/2), \alpha)}{(\rho/2, \alpha)}$$

where $(\ ,\ )$ denotes the Killing form and R^+ is the set of positive roots. Note that the above polynomial is of degree $\dim F$ and that $H(-\lambda - \rho) = (-1)^{\dim F} H(\lambda)$, which is Serre duality.

The Killing form allows to relate V to its dual. For every root $\alpha \in R$ we set $V^* \ni \alpha^\vee = (v \mapsto 2(\alpha, v)/(\alpha, \alpha))$. The facets of the Weyl chamber are supported by the simple roots, that is they are hypersurfaces defined by forms $\alpha^\vee$.

Lemma 7.1. *The extremal contraction $\hat{\pi}_\alpha : F \to F_\alpha$ associated to the facet $\alpha^\perp \cap$ $\mathrm{Nef}(F)$ is a $\mathbb{P}^1$ bundle and $\alpha^\vee$ is the class of the extremal curve in $N_1(F)$. The class of the relative cotangent bundle $\Omega(F/F_\alpha)$ in $\mathrm{Pic}\, F = \Lambda_W$ is $-\alpha$.*

Proof. Note that the restriction of the polynomial $H(\lambda)$ to this hyperplane $\alpha^\perp$ defined by $\alpha^\vee$ is of degree $\dim F - 1$ and $\alpha^\vee(\rho) = 2$, [TY05, 18.7.6]. This means that the extremal contraction $F \to F_\alpha$ associated to the facet $\alpha^\perp \cap \mathrm{Nef}(F)$ is a $\mathbb{P}^1$ bundle and $\alpha^\vee$ is the class of the fiber. On the other hand, $\rho - \alpha \in \alpha^\perp$ and $H(s_\alpha(\lambda)) - \alpha) = -H(\lambda)$ which is the relative duality □

Let X be the total space of the cotangent bundle of F, that is $X = \mathrm{Spec}_F(\mathrm{Symm}(TF))$. Recall that $TF = G \times_B \mathfrak{g}/\mathfrak{b}$, where $\mathfrak{b} \subset \mathfrak{g}$ is tangent to B and B acts on $\mathfrak{g}/\mathfrak{b}$ via adjoint representation and the quotient $\mathfrak{g} \to \mathfrak{g}/\mathfrak{b}$. Alternatively, $T^*F = G \times_B \mathfrak{u}$ where $\mathfrak{u} \subset \mathfrak{g}$ is the nilradical of $\mathfrak{b}$. The variety X is symplectic. Since TF is spanned by its global sections, the Lie algebra $\mathfrak{g}$, we have a map $X \to \mathfrak{g}^*$ which contracts the zero section to 0. The image is called the nilpotent cone which is a normal variety, we denote it by Y and $\pi : X \to Y$ is a symplectic contraction.
Clearly, $N^1(X/Y) = N^1(F)$, $\mathrm{Nef}(X/Y) = \mathrm{Nef}(F)$ and every extremal contraction $\hat{\pi}_\alpha : F \to F_\alpha$, which is a $\mathbb{P}^1$ bundle, extends to a divisorial contraction $\pi_\alpha : X \to X_\alpha$ with all nontrivial fibers being $\mathbb{P}^1$. Let $E_\alpha \subset X$ be the exceptional divisor of π_α and C_α be a general fiber of π_α restricted to E_α.

Lemma 7.2. *The class of C_α in $V^* = N_1(X/Y)$ is $\alpha^\vee$. The class of E_α in $\mathrm{Pic}\, X = \Lambda_W$ is $-\alpha$.*

Proof. We have an exact sequence of vector bundles over F:

$$0 \longrightarrow \hat{\pi}_\alpha^*(\Omega F_\alpha) \longrightarrow \Omega F \longrightarrow \Omega(F/F_\alpha) \longrightarrow 0$$

and the divisor E_α in the total space of ΩF is the total space of the sub-bundle $\pi_\alpha^*(\Omega F_\alpha)$. Thus, the restriction of its normal to F is the line bundle $\Omega(F/F_\alpha)$ hence the lemma follows by 7.1. □

Corollary 7.3. c.f. [Hin91, (5.2)] *In the above situation, the intersection matrix $E_\alpha \cdot C_\beta$ is the negative of the Cartan matrix of the respective root system.*

The above observation is the key for Brieskorn-Slodowy result on the type of codimension-2 singularity of the nilpotent cone which can be expressed as follows:

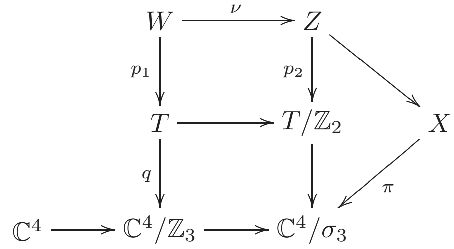
Theorem 7.4. (Brieskorn, Slodowy) *Let $\pi : X = G/B \to Y$ be the contraction to the nilpotent cone. If the root system of G is of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ then in codimension 2 the contraction π is the resolution of a surface Du Val singularity of the same $\mathbb{A} - \mathbb{D} - \mathbb{E}$ type. If G is of type $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{F}_4$ and $\mathbb{G}_2$ then in codimension 2 the contraction π is the resolution of singularities of type $\mathbb{A}_{2n-1}$, $\mathbb{D}_{n+1}$, $\mathbb{E}_6$ and $\mathbb{D}_4$ and the irreducible components of the exceptional set of π are in bijection with the orbits of the action of the group of automorphisms of the Dynkin diagrams of latter type.*

We have the following immediate consequence of 7.1 and 7.2.

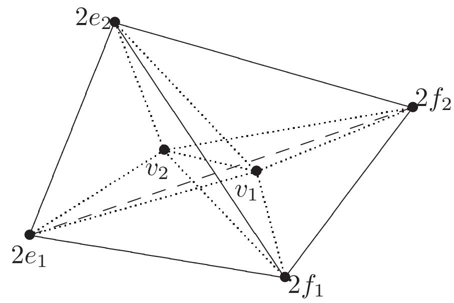
Corollary 7.5. *In the above case $\mathrm{Mov}(X/Y) = \mathrm{Nef}(X/Y)$ coincides with the Weyl chamber.*

7.2. **Resolving** $\mathbb{C}^4/\sigma_3$**.** We will give a description of the symplectic resolution of the quotient $\mathbb{C}^4/\sigma_3$. We refer to the following commutative diagram which comes from the presentation of $\sigma_3 = D_6$ in terms of a semisimple product $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$:

(7.2.13)



Here, $q : T \to \mathbb{C}^4/\mathbb{Z}_3$ is the toric resolution of $\mathbb{C}^4/\mathbb{Z}_3$ which can be described as follows: Let N_0 be a lattice with the basis e_1, e_2, f_1, f_2 and in $N_0 \otimes \mathbb{R}$ take the standard cone $\langle e_1, e_2, f_1, f_2\rangle$ representing $\mathbb{C}^4$. The toric singularity $\mathbb{C}^4/\mathbb{Z}_3$ is obtained by extending N_0 to an overlattice N (keeping the same cone) generated by adding to N_0 an extra generator $v_1 = (e_1 + e_2)/3 + 2(f_1 + f_2)/3$. If $v_2 = 2(e_1 + e_2)/3 + (f_1 + f_2)/3$ then the rays generated by e_i's, f_i's and v_i's are in the fan of the toric resolution of $\mathbb{C}^4/\mathbb{Z}_3$ which is presented in the following picture by taking a affine hyperplane section of the cone $\langle e_1, e_2, f_1, f_2\rangle$. The solid edges are the boundary of the cone while its division is marked by dotted line segments.



The exceptional set of this resolution consists of two divisors, E_1, E_2, both isomorphic to a $\mathbb{P}^2$- bundle over $\mathbb{P}^1$, namely $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O})$. They intersect along a smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

Coming back to the diagram 7.2.13: the action of $\mathbb{Z}_2$ on $\mathbb{C}^4/\mathbb{Z}_3$ can be lifted up to an action on T. This action, which is induced by the reflections in $\sigma_3 = D_6$, identifies the two divisors by identifying the $\mathbb{P}^2$ ruling of E_1 with this of E_2; it acts on the intersection by interchanging coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$.

Next, p_2 is the resolution of the quotient $T/\mathbb{Z}_2$ obtained by blowing up the surface which is the locus of A_1-singularities. The morphism p_1 is the blow-up along the fixed point set of the $\mathbb{Z}_2$-action. We denote by Δ_W and Δ_Z the exceptional divisors. Then ν is a $2:1$ cover ramified along Δ_W.

The divisor Δ_W is irreducible and its intersection with the fiber over the special point, which is the strict transform $E_1' \cup E_2'$, is equal to the 3rd Hirzebruch surface F_3. This follows from computing the normal of the curve which is the fixed point set of the $\mathbb{Z}_2$ action in the exceptional locus of T. Indeed, the normal of the intersection $E_1 \cap E_2 = \mathbb{P}^1 \times \mathbb{P}^1$ is $\mathcal{O}(1, -2) + \mathcal{O}(-2, 1)$ and the normal of the diagonal in the intersection is $\mathcal{O}(2)$. Thus the normal of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ in T is $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(2)$ and since its normal in the fixed point set is $\mathcal{O}(-1)$ it

follows that the normal of the fixed point set over the diagonal is $\mathcal{O}(-1) \oplus \mathcal{O}(2)$. Finally, let us note that the intersection in W of the F_3 surface with the strict transform of $\mathbb{P}^1 \times \mathbb{P}^1$ is the exceptional curve (section of the ruling) in the surface F_3 and the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$.
For $i = 1,\ 2$ fibers of the ruling $E_i \to \mathbb{P}^1$ are blown up in E_i' to ruled surfaces (1st Hirzebruch) and the map $E_i' \to \mathbb{P}^1$ can be factored either by blow down $E_i' \to E_i$ or by a $\mathbb{P}^1$-bundle $E_i' \to F_3$.
The strict trasform of the surface $E_1 \cap E_2 = \mathbb{P}^1 \times \mathbb{P}^1$ is mapped via the quotient map $W \to Z$ to $\mathbb{P}^2$, and this is a double covering ramified over the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$. The exceptional curve of F_3 which was diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ becomes a conic in $\mathbb{P}^2$. Thus, eventually, we see that E_1' is identified with E_2' to a (non-normal) divisor E_Z in Z. The divisors Δ_W and E_Z generate $PicZ$ and $K_Z = E_Z$.
From the computation of the intersection of curves and divisors we see that the divisor E_Z is not numerically effective hence Z admits birational Fano-Mori contraction $Z \to X$ with exceptional divisor E_Z. We describe the contraction by looking at the normalization of E_Z. Namely, by looking at the numerical classes of curves we conclude that the resulting map is a composition $E_1' \to F_3 \to S_3$ where the latter map is contraction of the exceptional curve in F_3 to the vertex of the cubic cone S_3. Therefore a general fiber of $Z \to X$ over E_Z is a $\mathbb{P}^1$ — that is, generally this is a blow-down of the divisor E_Z to a surface — while the special fiber is a $\mathbb{P}^2$. Such a contraction was discussed in [AW98] where it is proved that the image X is a smooth 4-fold and the divisor $E_Z \subset Z$ is blow-down to the rational cubic cone $S_3 \subset X$. Moreover $K_X = \mathcal{O}_X$.
Let us finally consider the induced map $\pi : X \to Y := \mathbb{C}^4/\sigma_3$. It is a crepant contraction which contracts the divisor Δ_W to a surface S which, outside the point 0, is a smooth surface of A_1 singularities (coming from the $\mathbb{Z}_2$-action); moreover it contracts S_3 to 0. The surface S is non-normal in 0. This is a crepant, hence symplectic, resolution of $\mathbb{C}^4/\sigma_3$.

Note that $Pic(X/Y) = \mathbb{Z}$, therefore $\mathrm{Mov}(X/Y)$ is one dimensional; this is the only SQM model over Y.

We conclude with the description of the family of rational curves (over Y). Let C be the essential curve of the symplectic resolution $\pi : X \to Y := \mathbb{C}^4/\sigma_3$ and let $\mathcal{V} \subset RatCurves^n(X/Y)$ be a family containing C. Then $\mathcal{V}$ is a smooth surface which contains a (-1)-curve, which parametrizes the lines in the ruling of S_3. The normalization of S is a smooth surface and $\mathcal{V}$ is obtained by blowing up the point of the normalization which stays over 0.

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