Induced Interactions for Ultra-Cold Fermi Gases in Optical Lattices

Dong-Hee Kim¹, Päivi Törmä¹, Jani-Petri Martikainen²

¹Department of Applied Physics, Helsinki University of Technology, Finland
²NORDITA, Sweden
What we want to study:

Superfluid transition of Fermi gases (Two component) (Attractive interaction)

in Optical Lattices (3D, 2D, …)

within the Mean-Field Theory (weak coupling regime)

beyond BCS predictions (many-body effect)

Medium-Induced Interaction

: Gorkov—Melik-Barkhudarov (GMB) correction
Medium effect in FREE gases

BCS result in vacuum:

\[ k_B T_c^{(0)} = \frac{8 \gamma}{\pi e^2} \epsilon_F \exp\left(-\frac{\pi}{2 k_F |a|}\right) \]

\[ U_0 = \frac{4 \pi \hbar}{m} a \]

Sâ de Melo et al., PRL 71, 3202 (1993).
Stoof et al., PRL 76, 10 (1996).

Correction due to a Medium effect:

Gorkov—Melik-Barkhudarov (GMB) correction

\[ \frac{T_c^{(0)}}{T_c} = (4e)^{1/3} \approx 2.22 \]

: 2\textsuperscript{nd} order correction to the interaction.

Heiselberg et al., PRL 85, 2418 (2000).
Superfluidity in optical lattices?

Indirect experimental evidence:
~ long range order?

Q. Medium effects in optical lattices?

The difference could be even more dramatic.

\[ T_c \propto \exp \left( -\frac{\pi}{2k_F|a|} \right) \quad : \text{correction changes the prefactor.} \]

\[ k_F|a| \rightarrow k_F|a|(1 + \delta k_F|a|) \quad : 2^{\text{nd}} \text{ order correction to interaction} \]

\[ T_c \rightarrow e^{\delta} T_c \quad : \text{varies sensitively with the correction} \]
Gorkov–Melik–Barkhudarov correction

Physical interpretation


Two fermions in the medium ≠ two fermions *in vacuo*.

Scattering in the medium:
Induced Interaction calculation

Induced interaction:

\[ U_{\text{ind}}(p, k) = -U_0^2 \int \frac{dq}{(2\pi)^D} \frac{f_{\uparrow, p+k+q} - f_{\downarrow, q}}{\xi_{\uparrow}(p + k + q) - \xi_{\downarrow}(q)} \]

Effective interaction:

Assume only the momenta at the Fermi surfaces contribute:

\[ \langle U_{\text{ind}} \rangle = \frac{1}{|S_{\uparrow}| |S_{\downarrow}|} \int_{S_{\uparrow}} ds_p \int_{S_{\downarrow}} ds_k U_{\text{ind}}(p, k) \]

\[ U_{\text{eff}} = U_0 + \langle U_{\text{ind}} \rangle \] : effective interaction replacing \( U_0 \)

Two interesting features

\[ \xi_{\sigma}(k) = \epsilon_{\sigma}(k) - \mu_{\sigma} \]
\[ f_{\sigma, k} = \frac{1}{1 + \exp[\beta \xi_{\sigma}(k)]} \]
1. Lindhard function (static) – susceptibility for non-interacting case

\[
U_{\text{ind}}(p, k) = -U_0^2 \int \frac{dq}{(2\pi)^D} \frac{f_{\uparrow, p+k+q} - f_{\downarrow, q}}{\xi_{\uparrow}(p + k + q) - \xi_{\downarrow}(q)}
\]

POSITIVE: screens the negative interatomic interaction \(U_0\)
suppresses \(T_C\) or the order parameter.

2. Perfect screening occurs at large interactions.

\[
U_{\text{eff}} = U_c + \langle U_{\text{ind}} \rangle = 0
\]

- parameter space of validity

At this large induced interaction, a RPA charge susceptibility can diverge.

\[
\chi_{RPA} = \frac{\chi_0}{1 + U_0 \chi_0} \to \infty
\]

- connection to other physics.

(Charge density waves)
**Mean-Field calculations**

Hubbard Hamiltonian:

\[
\mathcal{H} = -\sum_{\alpha} \sum_{i} t_{\alpha\sigma}(\hat{c}_{\sigma,i+}^{\dagger} \hat{c}_{\sigma,i} + \text{h.c.}) + U \sum_{i} \hat{n}_{i}^{\uparrow} \hat{n}_{i}^{\downarrow} - \mu \sum_{\sigma,i} \hat{n}_{\sigma i}
\]

Mean-Field approximation:

\[
\Delta = U \langle \hat{c}_{\uparrow i}^{\dagger} \hat{c}_{\downarrow i}^{\dagger} \rangle
\]

\[
\mathcal{H}_{MF} = \sum_{k} \left( \xi_{\uparrow k} \hat{c}_{\uparrow k}^{\dagger} \hat{c}_{\uparrow k} + \xi_{\downarrow k} \hat{c}_{\downarrow k}^{\dagger} \hat{c}_{\downarrow k} + \Delta \hat{c}_{\uparrow k}^{\dagger} \hat{c}_{\downarrow k} + \Delta \hat{c}_{\downarrow k} \hat{c}_{\uparrow k} \right) - \frac{\Delta^2}{U}
\]

\[
\epsilon_{\sigma}(k) = 2 \sum_{\alpha} t_{\sigma\alpha} [1 - \cos(k_{\alpha})]
\]

\[U \rightarrow U_0\] : the usual BCS calculations

\[U \rightarrow U_{\text{eff}}\] : the effective interaction with the GMB correction.

The zero temperature order parameter \(\Delta\) is calculated.
Minimization of Free Energy:

\[ \Omega = -\frac{\Delta^2}{U} + \int \frac{d\mathbf{k}}{(2\pi)^D} \left( \xi\downarrow(-\mathbf{k}) + E_-(\mathbf{k}) - \frac{1}{\beta} \ln(1 + e^{-\beta E_+(\mathbf{k})})(1 + e^{\beta E-(\mathbf{k})}) \right) \]

Effective interaction

\[ U_{\text{eff}} = U_0 + \langle U_{\text{ind}} \rangle \]

\[ E_{\pm}(\mathbf{k}) = \frac{\xi\uparrow(\mathbf{k}) - \xi\downarrow(-\mathbf{k})}{2} \pm \sqrt{\left( \frac{\xi\uparrow(\mathbf{k}) + \xi\downarrow(-\mathbf{k})}{2} \right)^2 + \Delta^2} \]

\[ \mu\uparrow = \mu\downarrow \quad : \text{Perfectly balanced gases are considered.} \]

Numerical Integrations: Monte Carlo algorithm is used.
The zero temperature order parameter $\Delta$ is calculated instead of $T_C$.

Both are equivalent in the weak coupling regime.
Induced interaction strength increases as chemical potential increases.

$$U_{\text{ind}}(p, k) = -U_0^2 \int \frac{d\mathbf{q}}{(2\pi)^D} \frac{f_{\uparrow, p+k+q} - f_{\downarrow, q}}{\xi_{\uparrow}(p + k + q) - \xi_{\downarrow}(q)}$$

$$(U_0 = -3t, T = 0)$$

$|U_0| < |U_c| \quad \forall \mu$$
The order parameter is suppressed much beyond the factor 2.22.

As the filling factor increases, correction effect becomes larger.

$$\frac{\Delta_{BCS}}{\Delta_{GMB}} \sim 2.22 \quad \text{at low density limit.}$$

$$\frac{\Delta_{BCS}}{\Delta_{GMB}} \sim 5 \quad \text{at } \mu = 4t.$$  

$$\frac{\Delta_{BCS}}{\Delta_{GMB}} \sim 25 \quad \text{at half filling.}$$

cf.) 1/D correction at high dimension

[van Dongen, PRL 67, 757 (1991)]

$$T_c^0 / T_c \sim O(1)$$

Department of Applied Physics, Helsinki University of Technology
2D Lattices: induced interaction

Induced interaction strength increases as chemical potential increases, but …

… induced interaction diverges at half filling.

\[ U_{\text{ind}}(p, k) = -U_0^2 \int \frac{dq}{(2\pi)^D} \frac{f_{\uparrow, p+k+q} - f_{\downarrow, q}}{\xi_{\uparrow}(p + k + q) - \xi_{\downarrow}(q)} \]

\[ \langle U_{\text{ind}} \rangle \sim \ln(4-\mu) \]

\( U_0 = -1.5t, T = 0 \)

Half-filling
2D Lattices: order parameter (T=0)

As μ increases, correction effect becomes larger.

- Effect of GMB correction

\[ \frac{\Delta_{BCS}}{\Delta_{GMB}} \sim 7 \quad \text{at } \mu = 3t. \]

- Comparisons with QMC

1. Decrease near half filling

2. Quantitative comparison

At quarter filling and \( U_0 = -4t \), \( k_B T_c \sim 0.05t \)

GMB correction: \( k_B T_c \sim 0.02t \)
The induced interaction diverges at half filling at $T=0$.

$$U_{\text{ind}}(p, k) = -U_0^2 \int \frac{d\mathbf{q}}{(2\pi)^D} \frac{f_{\uparrow, p+k+q} - f_{\downarrow, q}}{\xi_{\uparrow}(p + k + q) - \xi_{\downarrow}(q)}$$

Fermi surface nesting:

Divergence of the Lindhard function

Superfluidity and CDW coexist in the ground state at half filling in 2D lattices.
Lattice Anisotropy: crossover from 3D to 1D

Lattice anisotropy: \( \tilde{t} = \frac{t_y}{t_x} = \frac{t_z}{t_x} \)

1D limit: \( t_{y,z} \to 0 \)

Induced Interaction increases as the lattice goes toward quasi 1D.

Kinks are found at two points of changes in Fermi surface shapes.

\( \tilde{t} = 0.5 \)

The surface opens:
van Hove singularity

\( \tilde{t} = 0.25 \)

Quasi 1D shapes start to develop.

\( \tilde{t} \to 0 \) : Fermi surface nesting occurs.

Correction diverges.

\[
\begin{align*}
U_0 &= -3t \\
\mu &= 2t
\end{align*}
\]
Lattice Anisotropy: crossover from 3D to 1D

The order parameter decreases much beyond the BCS prediction.

However, the theory fails at 1D limit.

$U_{\text{ind}}$ diverges in 1D lattices.

$U_{\text{eff}} = 0$ at some point: $\Delta \rightarrow 0$

(incorrect!)

In quasi 1D lattices, finite gap exists.

(Larkin and Sak, PRL 1977; PRB 1978.)

$U_0 = -3t$

$\mu = 2t$
The presence of the optical lattice significantly enhances the effect of induced interactions on the BCS superfluidity.

\[ \Delta_{corr} \ll \Delta_{BCS} \]

The induced interaction correction extends the applicability of the mean-field calculations in lower dimensions.

- Predictions closer to QMC values in 2D lattices
- Divergence due to Fermi surface nesting: connection to different physics