Slice regular functions of several Clifford variables

R. Ghiloni and A. Perotti

Department of Mathematics, University of Trento, Italy, I-38123, Trento, Via Sommarive 14

Abstract. We introduce a class of slice regular functions of several Clifford variables. Our approach to the definition of slice functions is based on the concept of stem functions of several variables and on the introduction on real Clifford algebras of a family of commuting complex structures. The class of slice regular functions include, in particular, the family of (ordered) polynomials in several Clifford variables. We prove some basic properties of slice and slice regular functions and give examples to illustrate this function theory. In particular, we give integral representation formulas for slice regular functions and a Hartogs type extension result.

Keywords: Clifford algebras, slice regular functions, quaternions
PACS: 02.30.Fn, 02.10.Hh

INTRODUCTION

The concept of *slice regularity* for functions of *one quaternionic variable* has been introduced by Gentili and Struppa in [3, 4] and then extended to octonions in [5]. This function theory includes polynomials and power series in the quaternionic variable with quaternionic coefficients on one side. It can then be seen as an effective generalization to quaternions of the theory of holomorphic functions of one complex variable. A related theory, concerning *slice monogenic* functions on Clifford algebras, was introduced by Colombo, Sabadini and Struppa in [2]. In [6] and [7], a new approach to slice regularity, based on the concept of *stem function*, allowed to extend the theory to any real alternative *–algebra of finite dimension.

The original definition [3, 4] of slice regularity for a quaternionic function f, defined on an open domain Ω of $\mathbb{H}$, requires that, for every imaginary unit J, the restriction of f to the copy of $\mathbb{C}$ generated by J is holomorphic with respect to the complex structure defined by left multiplication by J. The approach taken in [6, 7] allows to embed the space of slice regular functions into a larger class, that of continuous *slice functions*, which corresponds in some sense to the usual complex continuous functions on the complex plane.

In the present paper, we propose a possible generalization of the theory to *several Clifford variables*. Our function theory includes, in particular, the class of (ordered) polynomials in several variables. Our approach is based on the concept of stem functions of several variables and on the introduction of a family of commuting complex structures on the real Clifford algebras. Several quaternionic variables have been studied recently also by Colombo, Sabadini and Struppa [1]. The approach via stem functions is similar to ours, but the definition of regularity is different, as we will see in the following.

After having given the basic definitions, we state some results which show the richness of this function theory. We state a Cauchy integral formula and some of its fundamental consequences, and we show that some results about the removability of singularities, which are true for several complex variables, continue to hold in this setting.

THE QUADRATIC CONE

Let $\mathbb{R}_m$ denote the real Clifford algebra of signature $(0,m)$ generated by $e_1,\dots,e_m$. Its elements can be expressed as

$$x=\sum_{K\in\mathscr{P}(m)} x_K e_K\ ,$$

where the sum is made over the subsets K of $\{1,\dots,m\}$, which can be identified with the increasing multiindices $(k_1,\dots,k_s)$, with $0\le s\le m$. The coefficients x_K are real numbers and the products $e_K=e_{k_1}\cdots e_{k_s}$ are the basis elements of the Clifford algebra $\mathbb{R}_m$. The unity of the Clifford algebra is obtained for $K=\emptyset$. As usual, we identify the real numbers field $\mathbb{R}$ with the subalgebra of $\mathbb{R}_m$ generated by the unity.

Let $\bar{x}$ denote the Clifford conjugate of $x\in\mathbb{R}_m$. Let $t(x)=x+\bar{x}$ be the *trace* of x and $n(x)=x\bar{x}$ the (squared) *norm* of a Clifford element x.

Definition 1. *The* quadratic cone *of* $\mathbb{R}_m$ *is the subset of* $\mathbb{R}_m$ *defined by*

$$\mathscr{Q}_m:=\mathbb{R}\cup\{x\in\mathbb{R}_m \mid t(x)\in\mathbb{R},\ n(x)\in\mathbb{R}, 4n(x)>t(x)^2\}\ .$$

Let $\mathbb{S}_m:=\{J\in\mathscr{Q}_m \mid J^2=-1\}=\{x\in\mathbb{R}_m \mid t(x)=0,\ n(x)=1\}$*. The elements of* $\mathbb{S}_m$ *are called the* square roots of -1 *in the Clifford algebra* $\mathbb{R}_m$.

The same definition can be given for any finite-dimensional, real alternative *-algebra (see [6, 7]), e.g.

for the real Clifford algebras $\mathbb{R}_{p,q}$ with signature (p,q). It can be easily seen that in $\mathbb{R}_m$ the last inequality in the definition of $\mathscr{Q}_m$ is automatically satisfied.

Proposition 1 ([6, 7])**.** *The quadratic cone $\mathscr{Q}_m$ satisfies the following properties:*

1. *$\mathscr{Q}_m$ is the whole algebra $\mathbb{R}_m$ only for $m = 1,2$, i.e. when $\mathbb{R}_m$ is $\mathbb{C}$ or $\mathbb{H}$.*
2. *If $m > 2$, $\mathscr{Q}_m$ is not a subalgebra or a subspace of $\mathbb{R}_m$.*
3. *$\mathscr{Q}_m$ contains the subspace of* paravectors
$$\mathbb{R}^{m+1} := \{x \in \mathbb{R}_m \mid [x]_k = 0 \text{ for every } k > 1\} ,$$
where $[x]_k$ denotes the k-vector part of x.
4. *$\mathscr{Q}_m$ is the real algebraic subset (proper for $m > 2$) of $\mathbb{R}_m$ defined by the equations*
$$x_K = 0,\ x \cdot (xe_K) = 0 \quad \forall e_K \neq 1 \text{ such that } e_K^2 = 1 ,$$
where $x \cdot y$ denotes the euclidean scalar product of $x,y \in \mathbb{R}_m \simeq \mathbb{R}^{2^m}$.

The simplest case when the quadratic cone is smaller than the full algebra $\mathbb{R}_m$ is the cone $\mathscr{Q}_3$, which is the six-dimensional real algebraic set

$$\mathscr{Q}_3 = \{x \in \mathbb{R}_3 \mid x_{123} = 0, x_1x_{23} - x_2x_{13} + x_3x_{12} = 0\} .$$

Proposition 2 ([6, 7])**.** *Let $Im(\mathbb{R}_m) := \{x \in \mathbb{R}_m \mid x^2 \in \mathbb{R},\ x \notin \mathbb{R} \setminus \{0\}\}$ be the set of purely imaginary elements of $\mathbb{R}_m$. Then the following properties hold.*

1. *For every $x \in \mathscr{Q}_m$, there exist unique $x_0 \in \mathbb{R}$, $y \in Im(\mathbb{R}_m) \cap \mathscr{Q}_m$, with $t(y) = 0$, such that*
$$x = x_0 + y .$$
2. *For every $J \in \mathbb{S}_m$, let $\mathbb{C}_J := \langle 1, J \rangle \simeq \mathbb{C}$ be the subalgebra generated by J. Then it holds:*
$$\mathscr{Q}_m = \bigcup_{J \in \mathbb{S}_m} \mathbb{C}_J$$
and $\mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R}$ for every $I, J \in \mathbb{S}_n$, $I \neq \pm J$.

SLICE REGULAR FUNCTIONS: ONE VARIABLE

We recall some definitions from [6, 7], where the slice regular functions of one variable in an alternative algebra were introduced.

Let $\mathbb{C}_m = \mathbb{R}_m \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified Clifford algebra, with representation

$$\mathbb{C}_m = \{w = x + iy \mid x, y \in \mathbb{R}_m,\ i^2 = -1\}$$

and complex conjugation $\overline{w} = \overline{x + iy} = x - iy$.

Definition 2. *Let $D \subseteq \mathbb{C}$. A function $F : D \to \mathbb{C}_m$ is called a* stem function *if it is* complex intrinsic*, i.e. $F(\overline{z}) = \overline{F(z)}$. Let*

$$\Omega_D := \{x = \alpha + \beta J \in \mathbb{C}_J \mid \alpha + i\beta \in D,\ J \in \mathbb{S}_m\}$$

be a circular set *in the quadratic cone $\mathscr{Q}_m$. Any stem function $F = F_1 + iF_2 : D \to \mathbb{C}_m$, with $\mathbb{R}_m$–valued components F_1, F_2, induces a (left)* slice function *$f : \Omega_D \to \mathbb{R}_m$. If $x = \alpha + \beta J \in D_J := \Omega_D \cap \mathbb{C}_J$, we set*

$$f(x) := F_1(z) + JF_2(z) \quad (z = \alpha + i\beta) .$$

Definition 3. *A slice function is* slice regular *if its stem function F is holomorphic.*

Examples 1.

1. Polynomials $p(x) = \sum_{j=0}^{d} x^j a_j$ *with right Clifford coefficients are slice regular functions on the quadratic cone $\mathscr{Q}_m$.*
2. Convergent power series $\sum_k x^k a_k$ *are slice regular functions on the intersection of $\mathscr{Q}_m$ with a ball centered in the origin.*
3. *A less elementary example is given by the quaternionic ($m = 2$) and Clifford ($m > 2$)* Joukowski transforms*:*
$$f(x) = \frac{1}{2}\left(x + \frac{1}{x}\right) ,$$
which are slice regular on $\mathscr{Q}_m \setminus \{0\}$.

If $m = 2$ and $D \cap \mathbb{R} \neq \emptyset$, then f is slice regular on Ω_D if and only if it is a *Cullen regular* function of a quaternionic variable, according to the definition given by Gentili and Struppa in [3, 4].

If $m > 2$ and $D \cap \mathbb{R} \neq \emptyset$, then f is slice regular on Ω_D if and only if the restriction of f to the paravectors is a *slice monogenic function*, a notion introduced by Colombo, Sabadini and Struppa in [2].

SLICE REGULAR FUNCTIONS: SEVERAL VARIABLES

Stem functions and slice functions

Let D be an open subset of $\mathbb{C}^n$, invariant w.r.t. complex conjugation in every variable $z_1, \dots, z_n$.

Definition 4. *A continuous function $F : D \to \mathbb{R}_m \otimes \mathbb{R}_n$, with $F = \sum_{K \in \mathscr{P}(n)} e_K F_K$, is called a* stem function *if it is* Clifford–intrinsic*, i.e. for each $K \in \mathscr{P}(n)$, $h \in \{1, \dots, n\}$*

and $z=(z_1,\dots,z_n)\in D$, *the components* $F_K : D \to \mathbb{R}_m$ *satisfy:*

$$F_K(z_1,\dots,z_{h-1},\bar z_h,z_{h+1},\dots,z_n)=\begin{cases} F_K(z) & \text{if } h\notin K,\\ -F_K(z) & \text{if } h\in K.\end{cases}$$

Let Ω_D be the *circular* subset of $(\mathcal{Q}_m)^n$ associated to $D\subseteq\mathbb{C}^n$:

$$\Omega_D=\{x=(x_1,\dots,x_n)\in(\mathcal{Q}_m)^n \mid x_h=\alpha_h+\beta_h J_h\in\mathbb{C}_J,\ J_h\in\mathbb{S}_m,(\alpha_1+i\beta_1,\dots,\alpha_n+i\beta_n)\in D\}\ .$$

Definition 5. *Given a stem function* $F : D\to\mathbb{R}_m\otimes\mathbb{R}_n$ *with* $F=\sum_{K\in\mathscr{P}(n)} e_K F_K$, *we define the* slice function $\mathscr{I}(F):\Omega_D\to\mathbb{R}_m$ *induced by* F *by setting*

$$\mathscr{I}(F)(x):=\textstyle\sum_{K\in\mathscr{P}(n)} J_K F_K(\alpha_1+i\beta_1,\dots,\alpha_n+i\beta_n)$$

for each $x=(x_1,\dots,x_n)=(\alpha_1+J_1\beta_1,\dots,\alpha_n+J_n\beta_n)$, *where* $J_K:=\prod^{\to}_{k\in K} J_k=J_{k_1}\cdots J_{k_s}$ *is the ordered product.*

We will denote by $\mathscr{S}(\Omega_D)$ *the real vector space of all slice functions on* Ω_D.

Examples 2.

1. *For each* $h=1,\dots,n$, *the* coordinate function x_h *is a slice function on* $(\mathcal{Q}_m)^n$*: if* $x_h=\alpha_h+J_h\beta_h$, *then* x_h *is induced by the stem function*

$$\zeta_h(z):=\alpha_h+e_h\beta_h\ .$$

2. *For each* $\ell\in\mathbb{N}^n$ *and* $a\in\mathbb{R}_m$, *the stem function*

$$\zeta_1^{\ell_1}(z)\cdots\zeta_n^{\ell_n}(z)a:=(\alpha_1+e_1\beta_1)^{\ell_1}\cdots(\alpha_n+e_n\beta_n)^{\ell_n}a$$

induces the monomial slice function $x^\ell a=\left(\prod^{\to}_{h\in\{1,\dots,n\}} x_h^{\ell_h}\right)a$ *on* $(\mathcal{Q}_m)^n$.

3. *Let* $L\subset\mathbb{N}^n$ *and* $a_\ell\in\mathbb{R}_m$ *for all* $\ell\in L$. *Then the* polynomial function *from* $(\mathcal{Q}_m)^n$ *to* $\mathbb{R}_m$, *sending* x *into* $p(x)=\sum_{\ell\in L}x^\ell a_\ell$, *is a slice function.*
4. Convergent power series $\sum_{\ell\in\mathbb{N}^n}x^\ell a_\ell$ *are slice functions on the intersection of* $(\mathcal{Q}_m)^n$ *with a product of balls centered in the origin.*

Proposition 3 (Smoothness)**.** *Let* $f=\mathscr{I}(F):\Omega_D\to\mathbb{R}_m$ *be a slice function. The following statements hold:*

1. *If* $F\in C^0(D,\mathbb{R}_m\otimes\mathbb{R}_n)$, *then* $f\in C^0(\Omega_D,\mathbb{R}_m)$.
2. *Let* $s_1=2^n(s+1)-1$. *If* $F\in C^{s_1}(D,\mathbb{R}_m\otimes\mathbb{R}_n)$ *for some* $s\in\mathbb{N}^*\cup\{\infty\}$, *then* $f\in C^s(\Omega_D,\mathbb{R}_m)$.
3. *If* $F\in C^\omega(D,\mathbb{R}_m\otimes\mathbb{R}_n)$, *then* $f\in C^\omega(\Omega_D,\mathbb{R}_m)$.

Let $f_I:\Omega_D\cap(\mathbb{C}_I)^n\to\mathbb{R}_m$ be the restriction of f on $\Omega_D\cap(\mathbb{C}_I)^n$.

Proposition 4 (Identity principle)**.** *Let* $f,g:\Omega_D\to\mathbb{R}_m$ *be slice functions and let* $I\in\mathbb{S}_m$ *such that* $f_I=g_I$. *Then* $f=g$ *on the whole domain* Ω_D.

Complex structures on $\mathbb{R}_n$

Let us introduce a family of n (almost) complex structures on $\mathbb{R}_n$.

Definition 6. *For each* $h=1,\dots,n$, *define the* complex structure $\mathscr{J}_h$ *on* $\mathbb{R}_n$ *by*

$$\mathscr{J}_h(e_K):=\begin{cases}-e_{K\setminus\{h\}} & \text{if } h\in K,\\ e_{K\cup\{h\}} & \text{if } h\notin K.\end{cases}$$

From the definition, it follows immediately that $\mathscr{J}_h^2=-id_{\mathbb{R}_n}$. In other words, the endomorphisms $\mathscr{J}_h$ are almost complex structures on $\mathbb{R}_n$. One can easily verify that $\mathscr{J}_1$ is the left multiplication by e_1, $\mathscr{J}_n$ is the right multiplication by e_n and, for every $h=1,\dots,n$, $\mathscr{J}_h$ coincides with the left multiplication by e_h on the complex plane $\mathbb{C}_{e_h}=\langle 1,e_h\rangle$ of $\mathbb{R}_n$. More generally, if $K\in\mathscr{P}(n)$ is such that $h\le k$ for every $k\in K$ then $\mathscr{J}_h(e_K)=e_he_K$.

Proposition 5. *The complex structures* $\mathscr{J}$ *are pairwise commuting and therefore they define commuting* Cauchy-Riemann operators *w.r.t.* $\mathscr{J}_h$*:*

$$\partial_h F=\frac12\left(\frac{\partial F}{\partial\alpha_h}-\mathscr{J}_h\left(\frac{\partial F}{\partial\beta_h}\right)\right),$$
$$\overline{\partial}_h F=\frac12\left(\frac{\partial F}{\partial\alpha_h}+\mathscr{J}_h\left(\frac{\partial F}{\partial\beta_h}\right)\right).$$

Slice regularity: several variables

We extend the complex structures $\mathscr{J}_h$ to $\mathbb{R}_m\otimes\mathbb{R}_n$ by setting

$$\mathscr{J}_h(a\otimes x)=a\otimes\mathscr{J}_h(x)$$

for every $a\in\mathbb{R}_m$, $x\in\mathbb{R}_n$.

Definition 7. *Let* $F:D\to\mathbb{R}_m\otimes\mathbb{R}_n$ *be a stem function of class* C^1 *and* $f=\mathscr{I}(F):\Omega_D\to\mathbb{R}_m$ *the induced slice function.* F *is called* holomorphic stem function *if, for each* $h=1,\dots,n$ *and each fixed* $z^0:=(z_1^0,\dots,z_n^0)$ *in* D, *the function* $F_h^{z^0}:D_h\to(\mathbb{R}_m\otimes\mathbb{R}_n,\mathscr{J}_h)$ *defined by* $z_h\mapsto F(z_1^0,\dots,z_{h-1}^0,z_h,z_{h+1}^0,\dots,z_n^0)$ *is holomorphic on a domain* D_h *of* $\mathbb{C}$ *containing* z_h^0. *Equivalently, if* $\overline{\partial}_hF=0$ *on* D *for every* $h=1,\dots,n$.

If F *is holomorphic, then we say that* $f=\mathscr{I}(F)$ *is a* slice regular function *on* Ω_D.

We will denote by $\mathscr{SR}(\Omega_D)$ the real vector space of slice regular function on Ω_D.

Remark 1. *For* $\mathbb{R}_2\simeq\mathbb{H}$, *several variables have been considered recently in [1]. Slice functions defined via stem functions are the same as ours, but regularity is different, since in [1] the complex structures* L_{e_h} *(left multiplication by* e_h*) are used in the place of* $\mathscr{J}_h$.

Examples 3.

1. *For each* $\ell \in \mathbb{N}^n$ *and* $a \in \mathbb{R}_m$*, the* monomial slice function $x^\ell a : (\mathcal{Q}_m)^n \to \mathbb{R}_m$ *defined for* $x = (x_1, \dots, x_n)$*,* $x_h = \alpha_h + J_h\beta_h$*, by*
$$x^\ell a = x_1^{\ell_1} \cdots x_n^{\ell_n} a = \mathscr{I}(\zeta_1^{\ell_1}(z) \cdots \zeta_n^{\ell_n}(z)a) = \\ = \mathscr{I}((\alpha_1 + e_1\beta_1)^{\ell_1} \cdots (\alpha_n + e_n\beta_n)^{\ell_n} a)$$
is regular. Therefore every (ordered) polynomial function $p(x) = \sum_{\ell \in L} x^\ell a_\ell$ *with right coefficients in* $\mathbb{R}_m$ *is slice regular.*
2. Convergent power series $\sum_{\ell \in \mathbb{N}^n} x^\ell a_\ell$ *are slice regular functions on the intersection of* $(\mathcal{Q}_m)^n$ *with a product of balls centered in the origin.*

Proposition 6. *Let* $F = \sum_{K \in P(n)} e_K F_K : D \to \mathbb{R}_m \otimes \mathbb{R}_n$ *be a stem function of class* C^1*. Let* $f = \mathscr{I}(F) : \Omega_D \to \mathbb{R}_m$*. We denote by* $f_I : \Omega_D \cap (\mathbb{C}_I)^n \to \mathbb{R}_m$ *the restriction of* f *on* $\Omega_D \cap (\mathbb{C}_I)^n$*. The following assertions are equivalent:*

1. f *is slice regular on* Ω_D*.*
2. $\dfrac{\partial F_K}{\partial \alpha_h} = \dfrac{\partial F_{K\cup\{h\}}}{\partial \beta_h}, \quad \dfrac{\partial F_K}{\partial \beta_h} = -\dfrac{\partial F_{K\cup\{h\}}}{\partial \alpha_h}$ *for each* K, h *with* $K \not\ni h$*.*
3. *There exists* $I \in \mathbb{S}_m$ *such that* f_I *is holomorphic w.r.t. the complex structures on* $(\mathbb{C}_I)^n$ *and on* $\mathbb{R}_m$ *defined by the left multiplication by* I*.*
4. *For all* $I \in \mathbb{S}_m$*,* f_I *is holomorphic w.r.t. the complex structures on* $(\mathbb{C}_I)^n$ *and on* $\mathbb{R}_m$ *defined by the left multiplication by* I*.*

As a consequence of Proposition 6, assuming $m = 1$ and $\mathbb{R}_1 \simeq \mathbb{C}$, we get that the definition of slice regularity for a function defined on a conjugation–invariant domain D of $\mathbb{C}^n$, is equivalent to the classical condition of holomorphicity of several complex variables.

Products and derivatives

A way to construct slice regular functions is to take the ordered product of stem functions of one variable.

Proposition 7. *Let* $D = \prod_{h=1}^n D_h$*. For each* $h = 1, \dots, n$*, let* $F^h : D_h \to \mathbb{R}_m \otimes \mathbb{C}_{e_h} \subseteq \mathbb{R}_m \otimes \mathbb{R}_n$ *be a (one variable) stem function of class* C^1*. Let* $F : D \to \mathbb{R}_m \otimes \mathbb{R}_n$ *defined by* $F(z_1, \dots, z_n) = \prod_{h \in \{1,\dots,n\}}^{\rightarrow} F^h(z_h)$*. Then* F *is a stem function, holomorphic if every* F^h *is holomorphic.*

In general, the *pointwise* product of two slice functions is *not* a slice function. However, the pointwise product of stem functions (in the algebra $\mathbb{R}_m \otimes \mathbb{R}_n$) is still a stem function.

Definition 8. *Let* $f = \mathscr{I}(F)$*,* $g = \mathscr{I}(G)$ *be slice functions on* Ω_D*. The* product *of* f *and* g *is the slice function* $f \cdot g := \mathscr{I}(FG)$*.*

The preceding definition is well posed, since the pointwise product FG of Clifford intrinsic functions is still Clifford intrinsic.

Let $K = \{k_1, \dots, k_p\} \in \mathscr{P}(n)$ be fixed, with $k_1 < \cdots < k_p$, and $F : D \to \mathbb{R}_m \otimes \mathbb{R}_n$ be a stem function, with $F = \sum_{K' \in \mathscr{P}(n)} e_{K'} F_{K'}$. We say that F is *K–reduced* if $F_{K'} = 0$ for each $K' \not\subset K$ and F depends only on $z_{k_1}, \dots, z_{k_p}$. Given $H \in \mathscr{P}(n)$, we write $K \le H$ if $k \le h$ for each $k \in K$ and for each $h \in H$.

Proposition 8. *If* $f = \mathscr{I}(F)$*,* $g = \mathscr{I}(G)$ *are slice regular functions,* F *is* K-reduced *and* G *is* H-reduced*, with* $K \le H$*, then* $f \cdot g$ *is slice regular.*

Remark 2. *The* ordering *of the variables is important for regularity: e.g. the function* $x_2 \cdot x_1 = \mathscr{I}(\zeta_2\zeta_1)$ *is a slice function but it is not slice regular.*

If $f = \mathscr{I}(F)$ is a slice function, of class C^1 on Ω_D, then also the functions $\partial_h F$ and $\overline{\partial}_h F$ are stem functions on D.

Definition 9. *We define the* slice derivatives *of* f *by setting*

$$\frac{\partial f}{\partial x_h} := \mathscr{I}(\partial_h F), \quad \frac{\partial f}{\partial \overline{x}_h} := \mathscr{I}\left(\overline{\partial}_h F\right), \quad h = 1, \dots, n.$$

These functions are continuous slice functions on Ω_D*.*

The slice function f is slice regular if and only if $\frac{\partial f}{\partial x_h^c} = 0$ for every $h = 1, \dots, n$. If f is slice regular, then also the derivatives $\frac{\partial f}{\partial x_h}$ are slice regular. This follows from the commutativity of the structures $\mathscr{J}_h$.

Cauchy integral formula

We now show that slice regular functions satisfy a Cauchy integral formula. As a consequence, we obtain that on a polydisc the class of slice regular functions coincides with that of convergent ordered power series.

Definition 10. *Let* $\Delta_y(x) := x^2 - t(y)x + n(y)$*,* $\Gamma_m := \{(x, y) \in \mathcal{Q}_m \times \mathcal{Q}_m \mid \Delta_y(x) \neq 0\}$*. The* Cauchy kernel *of* $\mathbb{R}_m$ *is the function* $C : \Gamma_m \to \mathbb{R}_m$*, slice regular in* x*, defined by*

$$C(x, y) := (\Delta_y(x))^{-1}(\bar{y} - x).$$

Fix $I \in \mathbb{S}_m$ and, for each $h = 1, \dots, n$, a bounded open subset E_h of $\mathbb{C}$, whose boundary is piecewise of class C^1. Let $E_h(I) := \Omega_{E_h} \cap \mathbb{C}_I$ and let $\partial E_h(I)$ be the boundary of $E_h(I)$ in $\mathbb{C}_I$. Let $E := E_1 \times \cdots \times E_n \subset \mathbb{C}^n$.

Denote by $\partial^* E(I)$ the *distinguished boundary* $\partial E_1(I) \times \cdots \times \partial E_n(I)$ of $E(I) := E_1(I) \times \cdots \times E_n(I)$.

Given $x=(x_1,\dots,x_n)$, $\xi=(\xi_1,\dots,\xi_n)\in(\mathbb{R}_m)^n$, let $C(x,\xi):=C(x_1,\xi_1)\cdots C(x_n,\xi_n)$.

Theorem 1 (Cauchy integral formula). *If f is a slice regular function on Ω_E, continuous on $\overline{\Omega}_E$, then, for all $x=(x_1,\dots,x_n)\in\Omega_E$, it holds:*

$$f(x)=\frac{1}{(2\pi)^n}\int_{\partial^*E(I)}C(x,\xi)\,d\xi_1\cdots d\xi_n I^{-n}f(\xi)\,.$$

In particular, f is real analytic.

Let $r=(r_1,\dots,r_n)\in(\mathbb{R}^+)^n$. Let $\|\cdot\|$ denote the euclidean norm of $\mathbb{R}_m\simeq\mathbb{R}^{2^m}$. Denote by B_r the polydisc $B(0,r_1)\times\cdots\times B(0,r_n)$ of $\mathbb{C}^n$ and by $B_m(r)$ the polydisc $B_m(r)=\{(x_1,\dots,x_n)\in(\mathbb{R}_m)^n\,|\,\|x_1\|<r_1,\dots,\|x_n\|<r_n\}$ of $(\mathbb{R}_m)^n$. Then $B_m(r)\cap(\mathscr{Q}_m)^n=\Omega_{B_r}$.

Corollary 1 (Ordered analyticity). *Let f be a slice regular function on Ω_{B_r}, continuous on $\overline{\Omega}_{B_r}$. Choose $I\in\mathbb{S}_m$ and, for each $\ell=(\ell_1,\dots,\ell_n)\in\mathbb{N}^n$, define $a_\ell\in\mathbb{R}_m$ by setting*

$$a_\ell:=(2\pi I)^{-n}\int_{\partial^*B_r(I)}\xi_1^{-\ell_1-1}\cdots\xi_n^{-\ell_n-1}d\xi_1\cdots d\xi_n f(\xi).$$

Then the ordered power series $\sum_{\ell\in\mathbb{N}^n}x_1^{\ell_1}\cdots x_n^{\ell_n}a_\ell$ converges uniformly on compact subsets of $B_m(r)$ and its sum is equal to $f(x)$ for each $x\in\Omega_{B_r}$.

Corollary 2. *On Ω_{B_r}, the set of slice regular functions coincides with the one of convergent ordered power series.*

Corollary 3 (Cauchy's inequalities). *Let f be a slice regular function on Ω_{B_r}, continuous on $\overline{\Omega}_{B_r}$ and let M be a constant such that $\sup_{x\in\partial^*B_r(I)}\|f(x)\|\le M$ for some $I\in\mathbb{S}_m$. Then there exists a constant N_m such that, for each $\ell=(\ell_1,\dots,\ell_n)\in\mathbb{N}^n$, it holds:*

$$\|\partial_\ell f(0)\|\le N_m\cdot M\cdot\ell!\cdot r_1^{-\ell_1}\cdots r_n^{-\ell_n},$$

where $\partial_\ell:=\partial^{\ell_1+\cdots+\ell_n}/\partial Re(x_1)^{\ell_1}\cdots\partial Re(x_n)^{\ell_n}$.

Removability of singularities

Using algebraic analysis on the system of real partial differential equations corresponding to

$$\overline{\partial}_h F=0,\quad h=1,\dots,n,\quad \text{with } f=\mathscr{I}(F),$$

thanks to the commutativity of the Cauchy-Riemann operators $\overline{\partial}_h$, it is possible to prove a Hartogs type extension result. It holds under the same topological assumptions on the domain as in the several complex variables case.

Theorem 2 (Hartogs extension phenomenon). *Let $D'\subset D\subset\mathbb{C}^n$ be an open set with compact closure $K:=\overline{D'}\subset D$ such that $D\setminus K$ is connected. If f is a slice regular function on $\Omega_{D\setminus K}=\Omega_D\setminus\overline{\Omega}_{D'}$, then it extends uniquely to a slice regular function on the whole set Ω_D.*

Another result about removability of singularities is a generalization of Riemann's theorem.

Theorem 3. *Let Θ be a circular open subset of $(\mathbb{R}_m)^n$, let $Z=\Omega_W$ be a proper closed subset of Θ with W locally analytic in $\mathbb{C}^n$ and let f be a slice regular function on $\Theta\setminus Z$. Suppose that at least one of the following two condition holds: (1) f is locally bounded in Θ, (2) $\mathrm{codim}(W)\ge 2$. Then f extends to a slice regular function on the whole Θ.*

ACKNOWLEDGMENTS

The present work was partially supported by GNSAGA of INdAM.

REFERENCES

1. F. Colombo, I. Sabadini, and D. C. Struppa. *Algebraic properties of the module of slice regular functions in several quaternionic variables*. Preprint.
2. F. Colombo, I. Sabadini, and D. C. Struppa. *Slice monogenic functions.* Israel J. Math., **171** (2009), 385–403.
3. G. Gentili and D. C. Struppa. *A new approach to Cullen-regular functions of a quaternionic variable.* C. R. Math. Acad. Sci. Paris, **342** (10) (2006), 741–744.
4. G. Gentili and D. C. Struppa. *A new theory of regular functions of a quaternionic variable.* Adv. Math., **216** (1) (2007), 279–301.
5. G. Gentili and D. C. Struppa. *Regular functions on the space of Cayley numbers.* Rocky Mountain J. Math., **40** (2010), no. 1, 225–241.
6. R. Ghiloni and A. Perotti. *Slice regular functions on real alternative algebras.* Adv. Math., **226** (2) (2011), 1662–1691.
7. R. Ghiloni and A. Perotti. *A New Approach to Slice regularity on Real Algebras.* In "Hypercomplex Analysis and its Applications", I. Sabadini and F. Sommen (eds.), Trends Math., Birkhäuser, Basel, 2011, 109–124.