ON REGULAR HARMONICS OF ONE QUATERNIONIC VARIABLE

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Abstract. We prove some results about the Fueter-regular homogeneous polynomials, which appear as components in the power series of any quaternionic regular function. We obtain a differential condition that characterizes the homogeneous polynomials whose trace on the unit sphere extends as a regular polynomial. We apply this result to define an injective linear operator from the space of complex spherical harmonics to the module of regular homogeneous polynomials of a fixed degree $k$.

1. Introduction

Let $B$ denote the unit ball in $\mathbb{C}^2 \simeq \mathbb{H}$ and $S = \partial B$ the group of unit quaternions. In §3.1 we obtain a differential condition that characterizes the homogeneous polynomials whose restriction to $S$ coincides with the restriction of a regular polynomial. This result generalizes a similar characterization for holomorphic extensions of polynomials proved by Kytmanov (cf. [2] and [3]).

In §3.2 we show how to define an injective linear operator $R : \mathcal{H}_k(S) \rightarrow U^\psi_k$ from the space $\mathcal{H}_k(S)$ of complex-valued spherical harmonics of degree $k$ to the $\mathbb{H}$-module $U^\psi_k$ of $\psi$-regular homogeneous polynomials of the same degree (cf. §2.2 and §3.2 for precise definitions). In particular, we show how to construct bases of the module of regular homogeneous polynomials of a fixed degree starting from any choice of $\mathbb{C}$-bases of the spaces of complex harmonic homogeneous polynomials.

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2. Notations and definitions

2.1. Let $\Omega = \{ z \in \mathbb{C}^2 : \rho(z) < 0 \}$ be a bounded domain in $\mathbb{C}^2$ with smooth boundary. Let $\nu$ denote the outer unit normal to $\partial \Omega$ and $\tau = i\nu$. For every $F \in C^1(\Omega)$, let $\overline{\partial}_n F = \frac{1}{2} \left( \frac{\partial F}{\partial \nu} + i \frac{\partial F}{\partial \tau} \right)$ be the normal component of $\overline{\partial} F$ (see Kytmanov [2]§§3.3 and 14.2). It can be expressed by means of the Hodge $\ast$-operator and the Lebesgue surface measure as $\overline{\partial}_n f d\sigma = \ast \overline{\partial} F |_{\partial \Omega}$. In a neighbourhood of $\partial \Omega$ we have the decomposition of $\overline{\partial} F$ in the tangential and the normal parts:

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\[ \partial F = \overline{\partial} F + \mathrm{d} \wedge \overline{\partial} F. \]

We denote by \( L \) the tangential Cauchy-Riemann operator
\[ L = \frac{1}{\partial \rho} \left( \frac{\partial}{\partial z} \sigma_{12} + \frac{\partial}{\partial z} \sigma_{13} \right). \]

Let \( \mathbb{H} \) be the algebra of quaternions \( q = x_0 + ix_1 + jx_2 + kx_3 \), where \( x_0, x_1, x_2, x_3 \) are real numbers and \( i, j, k \) denote the basic quaternions. We identify the space \( \mathbb{C}^2 \) with the set \( \mathbb{H} \) by means of the mapping that associates the quaternion \( q = z_1 + z_2j \) to \((z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)\). We refer to Sudbery [8] for the basic facts of quaternionic analysis. We will denote by \( D \) the left Cauchy-Riemann-Fueter operator
\[ D = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3}. \]

A quaternionic \( C^1 \) function \( f = f_1 + f_2j \), is \( \text{(left-)regular} \) on a domain \( \Omega \subseteq \mathbb{H} \) if \( Df = 0 \) on \( \Omega \). We prefer to work with another class of regular functions, which is more explicitly connected with the hyperkähler structure of \( \mathbb{H} \). It is defined by the Cauchy-Riemann-Fueter operator associated to the structural vector \( \psi = \{1, i, j, -k\} \):
\[ D' = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} - k \frac{\partial}{\partial x_3} = 2 \left( \frac{\partial}{\partial z_1} + j \frac{\partial}{\partial z_2} \right). \]

A quaternionic \( C^1 \) function \( f = f_1 + f_2j \), is called \( \text{(left-)}\psi\text{-regular} \) on a domain \( \Omega \), if \( D'f = 0 \) on \( \Omega \). This condition is equivalent to the following system of complex differential equations:
\[ \frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \quad \frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1}. \]

The identity mapping is \( \psi \)-regular, and any holomorphic mapping \((f_1, f_2)\) on \( \Omega \) defines a \( \psi \)-regular function \( f = f_1 + f_2j \). This is no more true if we replace \( \psi \)-regularity with regularity. Moreover, the complex components of a \( \psi \)-regular function are either both holomorphic or both non-holomorphic (cf. Vasilevski [9], Mitelman et al [4] and Perotti [5]). Let \( \gamma \) be the transformation of \( \mathbb{C}^2 \) defined by \( \gamma(z_1, z_2) = (z_1, z_2) \). Then a \( C^1 \) function \( f \) is regular on the domain \( \Omega \) if, and only if, \( f \circ \gamma \) is \( \psi \)-regular on \( \gamma^{-1}(\Omega) \).

2.2. The two-dimensional Bochner-Martinelli form \( U(\zeta, z) \) is the first complex component of the Cauchy-Fueter kernel \( G'(p - q) \) associated to \( \psi \)-regular functions (cf. Fueter [1], Vasilevski [9], Mitelman et al [4]). Let \( q = z_1 + z_2j, p = \zeta_1 + \zeta_2j, \sigma(q) = dq[0] - i dx[1] + j dx[2] + kdx[3] \), where \( dx[k] \) denotes the product of \( dx_0, dx_1, dx_2, dx_3 \) with \( dx_k \) deleted. Then \( G'(p - q) \sigma(p) = U(\zeta, z) + \omega(\zeta, z)j \), where \( \omega(\zeta, z) \) is the complex \((1, 2)\)-form
\[ \omega(\zeta, z) = -\frac{1}{4\pi^2} |\zeta - z|^{-4}((\overline{\zeta} - \overline{z})d\zeta_1 + (\overline{\zeta} - \overline{z})d\zeta_2) \wedge \overline{\zeta}. \]

Here \( \overline{\zeta} = \overline{\zeta_1} \wedge \overline{\zeta_2} \) and we choose the orientation of \( \mathbb{C}^2 \) given by the volume form \( \frac{1}{4} dz_1 \wedge dz_2 \wedge dx_1 \wedge dx_2 \). Given \( g(\zeta, z) = \frac{1}{4\pi^2} |\zeta - z|^{-2} \), we can also write \( U(\zeta, z) = -2 \ast \partial_\zeta g(\zeta, z) \) and \( \omega(\zeta, z) = -\partial_\zeta (g(\zeta, z)\overline{\zeta}). \)
3. Regular polynomials

3.1. In this section we will obtain a differential condition that characterizes the homogeneous polynomials whose restrictions to the unit sphere extend regularly or \( \psi \)-regularly. We will use a computation made by Kytmanov in [3] (cf. also [2] Corollary 23.4), where the analogous result for holomorphic extensions is proved.

Let \( \Omega \) be the unit ball \( B \) in \( \mathbb{C}^2 \), \( S = \partial B \) the unit sphere. In this case the operators \( \overline{\partial}_n \) and \( L \) have the following forms:

\[
\overline{\partial}_n = \bar{z}_1 \frac{\partial}{\partial z_1} + \bar{z}_2 \frac{\partial}{\partial z_2}, \quad L = z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2}
\]

and they preserve harmonicity. Let \( \Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} \) be the Laplacian in \( \mathbb{C}^2 \) and \( D_k \) the differential operator

\[
D_k = \sum_{0 \leq l \leq k/2-1} \frac{(k - 2l - 1)!(2l - 1)!}{k!(l + 1)!} 2^l \Delta^{l+1}.
\]

**Theorem 1.** Let \( f = f_1 + f_2 j \) be a \( \mathbb{H} \)-valued, homogeneous polynomial of degree \( k \). Then its restriction to \( S \) extends as a \( \psi \)-regular function into \( B \) if, and only if,

\[
(\overline{\partial}_n - D_k) f_1 + L(f_2) = 0 \quad \text{on } S.
\]

**Proof.** In the first part we can proceed as in [3]. The harmonic extension \( \tilde{f}_1 \) of \( f_1 \) into \( B \) is given by Gauss’s formula: \( \tilde{f}_1 = \sum_{s \geq 0} g_{k-2s} \), where \( g_{k-2s} \) is the homogeneous harmonic polynomial of degree \( k - 2s \) defined by

\[
g_{k-2s} = \frac{k - 2s + 1}{s!(k - s + 1)!} \sum_{j \geq 0} \frac{(-1)^j (k - j - 2s)!}{j!} |z|^{2j} \Delta^{j+s} f_1.
\]

Then \( \overline{\partial}_n \tilde{f}_1 = \overline{\partial}_n f_1 - D_k f_1 \) on \( S \) (cf. [2] §23). Let \( \tilde{f}_2 \) be the harmonic extension of \( f_2 \) into \( B \) and \( \tilde{f} = \tilde{f}_1 + \tilde{f}_2 j \). Then \( (\overline{\partial}_n - D_k) f_1 + L(f_2) = 0 \) on \( S \) is equivalent to \( \overline{\partial}_n \tilde{f}_1 + L(f_2) = 0 \) on \( S \). We now show that this implies the \( \psi \)-regularity of \( \tilde{f} \). Let \( F^+ \) and \( F^- \) be the \( \psi \)-regular functions defined respectively on \( B \) and on \( \mathbb{C}^2 \setminus \overline{B} \) by the Cauchy-Fueter integral of \( \tilde{f} \):

\[
F^\pm(z) = \int_S U(\zeta, z) \tilde{f}(\zeta) + \int_S \omega(\zeta, z) j \tilde{f}(\zeta).
\]

From the equalities \( U(\zeta, z) = -2 \ast \partial_\zeta g(\zeta, z), \omega(\zeta, z) = -\partial_\zeta (g(\zeta, z) d\zeta) \), we get that

\[
F^+(z) = -2 \int_S (\tilde{f}_1(\zeta) + f_2(\zeta) j) \ast \partial_\zeta g(\zeta, z) - \int_S \partial_\zeta (g(\zeta, z) d\zeta)(\overline{\tilde{f}_1} j - \overline{f_2})
\]

for every \( z \notin \overline{B} \). From the complex Green formula and Stokes’ Theorem and from the equality \( \overline{\partial}_n f_1 \wedge d\zeta|_S = 2L(f_2) d\sigma \) on \( S \), we get that the first complex
component of $F^-(z)$ is
\[ -2\int_S f \bar{\partial}_n g d\sigma + \int_S f_2 \bar{\partial}_2 g \wedge d\bar{c} = -2\int_S g \bar{\partial}_n f_1 d\sigma - \int_S g \partial_\xi f_2 \wedge d\bar{c} \]
\[ = -2\int_S g(\bar{\partial}_n f_1 + L(f_2)) d\sigma \]
and then it vanishes on $C^2 \setminus \mathcal{B}$. Therefore, $F^- = F_{2j}$, with $F_2$ a holomorphic function that can be holomorphically continued to the whole space. Let $\tilde{F}^- = \tilde{F}_{2j}$ be such extension. Then $F = F^+ - \tilde{F}^\mathcal{B}$ is a $\psi$-regular function on $\mathcal{B}$ (indeed a polynomial of the same degree $k$), continuous on $\mathcal{B}$, such that $F|_S = f|_S$. The converse is immediate from the equations of $\psi$-regularity.

Let $N$ and $T$ be the differential operators
\[ N = \bar{z}_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}, \quad T = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}. \]

$T$ is a tangential operator w.r.t. $S$, while $N$ is non-tangential, such that $N(\rho) = |\bar{\partial}|^2 \rho$, $\text{Re}(N) = |\partial| \text{Re}(\bar{\partial}_n)$, where $\rho = |z_1|^2 + |z_2|^2 - 1$. Let $\gamma$ be the reflection introduced at the end of §1.1. The operator $D_\gamma$ is $\gamma$-invariant, i.e. $D_\gamma(f \circ \gamma) = D_\gamma(f) \circ \gamma$, since $\Delta$ is invariant. It follows a criterion for regularity of homogeneous polynomials.

**Corollary 2.** Let $f = f_1 + f_{2j}$ be a $\mathbb{H}$-valued, homogeneous polynomial of degree $k$. Then its restriction to $S$ extends as a regular function into $\mathcal{B}$ if, and only if,
\[ (N - D_\gamma)f_1 + T(f_2) = 0 \quad \text{on } S. \]

Let $g = \sum_k g^k$ be the homogeneous decomposition of a polynomial $g$. After replacing $D_k g$ by $\sum_k D_k g^k$, we can extend the preceding results also to non-homogeneous polynomials.

**3.2.** Let $\mathcal{P}_k$ denote the space of homogeneous complex-valued polynomials of degree $k$ on $C^2$, and $\mathcal{H}_k$ the space of harmonic polynomials in $\mathcal{P}_k$. The space $\mathcal{H}_k$ is the sum of the pairwise $L^2(S)$-orthogonal spaces $\mathcal{H}_{k,p}$ ($p + q = k$), whose elements are the harmonic homogeneous polynomials of degree $p$ in $z_1, z_2$ and $q$ in $\bar{z}_1, \bar{z}_2$ (cf. for example Rudin [7], §12.2). The spaces $\mathcal{H}_k$ and $\mathcal{H}_{p,q}$ can be identified with the spaces of the restrictions of their elements to $S$ (spherical harmonics). These spaces will be denoted by $\mathcal{H}_k(S)$ and $\mathcal{H}_{p,q}(S)$ respectively.

Let $U^\psi_k$ be the right $\mathbb{H}$-module of (left)$\psi$-regular homogeneous polynomials of degree $k$. The elements of the modules $U^\psi_k$ can be identified with their restrictions to $S$, which we will call regular harmonics.

**Theorem 3.** For every $f_1 \in \mathcal{P}_k$, there exists $f_2 \in \mathcal{P}_k$ such that the trace of $f = f_1 + f_{2j}$ on $S$ extends as a $\psi$-regular polynomial of degree at most $k$ on $\mathbb{H}$. If $f_1 \in \mathcal{H}_k$, then $f_2 \in \mathcal{H}_k$ and $f = f_1 + f_{2j} \in U^\psi_k$. 
Proof. We can suppose that $f_1$ has degree $p$ in $z$ and $q$ in $\bar{z}$, $p + q = k$, and then extend by linearity. Let $\tilde{f}_1 = \sum_{s \geq 0} g_{p-s,q-s}$ be the harmonic extension of $f_1$ into $B$, where $g_{p-s,q-s} \in \mathcal{H}_{p-s,q-s}$ is given by formula (*). Then $\overline{\partial}_n L(g_{p-s,q-s}) = (p - s + 1)L(g_{p-s,q-s})$. We set
\[
\tilde{f}_2 = \sum_{s \geq 0} \frac{1}{p - s + 1} L(g_{p-s,q-s}) \in \bigoplus_{s \geq 0} \mathcal{H}_{k-2s}.
\]
Then $\overline{\partial}_n \tilde{f}_2 = \overline{L(f_1)}$ on $S$ and we can conclude as in the proof of Theorem 1 that $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 j$ is a $\psi$-regular polynomial of degree at most $k$. Now it suffices to define
\[
f_2 = \sum_{s \geq 0} \frac{|z|^{2s}}{p - s + 1} L(g_{p-s,q-s}) \in \mathcal{P}_k
\]
to get a homogeneous polynomial $f = f_1 + f_2 j$, of degree $k$, that has the same restriction to $S$ as $\tilde{f}$. If $f_1 \in \mathcal{H}_k$, then $\tilde{f}_1 = f_1$, $\tilde{f}_2 = f_2$ and therefore $f \in U^\psi_k$. □

Let $C : U^\psi_k \to \mathcal{H}_k(S)$ be the complex-linear operator that associates to $f = f_1 + f_2 j$ the restriction to $S$ of its first complex component $f_1$. The function $\tilde{f}$ in the preceding proof gives a right inverse $R : \mathcal{H}_k(S) \to U^\psi_k$ of the operator $C$. The function $R(f_1)$ is uniquely determined by the orthogonality condition with respect to the functions holomorphic on a neighbourhood of $B$:
\[
\int_S (R(f_1) - f_1) \bar{h} d\sigma = 0 \quad \forall h \in O(B).
\]

**Corollary 4.** (i) The restriction operator $C$ defined on $U^\psi_k$ induces isomorphisms of real vector spaces
\[
U^\psi_k / \mathcal{H}_{k,0} \cong \mathcal{H}_k(S), \quad U^\psi_k / \mathcal{H}_{k,0} + \mathcal{H}_{k,0} \cong \mathcal{H}_k(S) / \mathcal{H}_{k,0}(S).
\]
(ii) $U^\psi_k$ has dimension $\frac{1}{2}(k + 1)(k + 2)$ over $\mathbb{H}$.

Proof. The first part follows from $ker C = \{ f = f_1 + f_2 j : f_1 = 0 \text{ on } S \} = \mathcal{H}_{k,0}$. Part (ii) can be obtained from any of the above isomorphisms, since $\mathcal{H}_{k,0}$ (as every space $\mathcal{H}_{p,q}, p + q = k$) and $\mathcal{H}_k(S)$ have real dimensions respectively $2(k + 1)$ and $2(k + 1)^2$. □

As an application of Corollary 2, we have another proof of the known result (cf. Sudbery [8] Theorem 7) that the right $\mathbb{H}$-module $U_k$ of left-regular homogeneous polynomials of degree $k$ has dimension $\frac{1}{2}(k + 1)(k + 2)$ over $\mathbb{H}$. 
3.3. The operator $R : \mathcal{H}_k(S) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(S) \rightarrow U_k^\psi$ can also be used to obtain H-bases for $U_k^\psi$ starting from bases of the complex spaces $\mathcal{H}_{p,q}(S)$. On $\mathcal{H}_{p,q}(S)$, $R$ acts in the following way:

$$R(h) = h + M(h)j, \text{ where } M(h) = \frac{1}{p+1}L(h) \in \mathcal{H}_{q-1,p+1} \ (h \in \mathcal{H}_{p,q})$$

Note that $M \equiv 0$ on $\mathcal{H}_{k,0}(S)$. If $q > 0$, $M^2 = -Id$ on $\mathcal{H}_{p,q}(S)$, since $qh = \overline{\partial_n}h = -L(M(h))$ on $S$, and therefore

$$h = -\frac{1}{q}L(M(h)) = -\frac{1}{q(p+1)}LL(h) = -M^2(h).$$

If $k = 2m+1$ is odd, then $M$ is a complex conjugate isomorphism of $\mathcal{H}_{m,m+1}(S)$. Then $M$ induces a quaternionic structure on this space, which has real dimension $4(m+1)$. We can find complex bases of $\mathcal{H}_{m,m+1}(S)$ of the form

$$\{h_1, M(h_1), \ldots, h_{m+1}, M(h_{m+1})\}.$$  

**Theorem 5.** Let $B_{p,q}$ denote a complex base of the space $\mathcal{H}_{p,q}(S)$ ($p + q = k$). Then:

(i) if $k = 2m$ is even, a basis of $U_k^\psi$ over $\mathbb{H}$ is given by the set

$$B_k = \{R(h) : h \in B_{p,q}, p + q = k, 0 \leq q \leq p \leq k\}.$$  

(ii) if $k = 2m + 1$ is odd, a basis of $U_k^\psi$ over $\mathbb{H}$ is given by

$$B_k = \{R(h) : h \in B_{p,q}, p + q = k, 0 \leq q < p \leq k\} \cup \{R(h_1), \ldots, R(h_{m+1})\},$$

where $h_1, \ldots, h_{m+1}$ are chosen such that the set

$$\{h_1, M(h_1), \ldots, h_{m+1}, M(h_{m+1})\}$$

forms a complex basis of $\mathcal{H}_{m,m+1}(S)$.

If the bases $B_{p,q}$ are orthogonal in $L^2(S)$ and $h_1, \ldots, h_{m+1} \in \mathcal{H}_{m,m+1}(S)$ are mutually orthogonal, then $B_k$ is orthogonal, with norms

$$\|R(h)\|_{L^2(S, \mathbb{H})} = \left(\frac{p + q + 1}{p + 1}\right)^{1/2} \|h\|_{L^2(S)} \ (h \in B_{p,q})$$

w.r.t. the scalar product of $L^2(S, \mathbb{H})$.

**Proof.** From dimension count, it suffices to prove that the sets $B_k$ are linearly independent. When $q \leq p$, $q' \leq p'$, $p + q = p' + q' = k$, the spaces $\mathcal{H}_{p,q}$ and $\mathcal{H}_{q'-1,p'+1}$ are distinct. Since $R(h) = h + M(h)j \in \mathcal{H}_{p,q} \oplus \mathcal{H}_{q-1,p+1}$, this implies the independence over $\mathbb{H}$ of the images $\{R(h) : h \in B_{p,q}\}$. It remains to consider the case when $k = 2m + 1$ is odd. If $h \in \mathcal{H}_{m,m+1}(S)$, the complex components $h$ and $M(h)$ of $R(h)$ belong to the same space. The independence of $\{R(h_1), \ldots, R(h_{m+1})\}$ over $\mathbb{H}$ follows from the particular form of the complex basis chosen in $\mathcal{H}_{m,m+1}(S)$.
The scalar product of $L(h)$ and $L(h')$ in $\mathcal{H}_{p,q}(S)$ is

$$(L(h), L(h')) = (h, L^*(L(h'))) = q(p+1)(h, h'),$$

since the adjoint $L^*$ is equal to $-\bar{T}$ (cf. [7], §18.2.2) and $T L = q(p+1)M^2 = -q(p+1)I_d$. Therefore, if $h, h'$ are orthogonal, $M(h)$ and $M(h')$ are orthogonal in $\mathcal{H}_{q-1,p+1}$ and then also $R(h)$ and $R(h')$. Finally, the norm of $R(h)$, $h \in \mathcal{H}_{p,q}(S)$, is

$$\|R(h)\|^2 = \|h\|^2 + |M(h)|^2 = \|h\|^2 + \frac{1}{(p+1)^2}\|L(h)\|^2 = \frac{p+q+1}{p+1}\|h\|^2$$

and this concludes the proof. \(\square\)

From Theorem 3 it is immediate to obtain also bases of the right $\mathbb{H}$-module $U_k$ of left-regular homogeneous polynomials of degree $k$.

**Examples.** (i) The case $k = 2$. Starting from the orthogonal bases $B_{2,0} = \{z_1^2, 2z_1z_2, z_2^2\}$ of $\mathcal{H}_{2,0}$ and $B_{1,1} = \{z_1z_2, |z_1|^2 - |z_2|^2, z_2z_1\}$ of $\mathcal{H}_{1,1}$ we get the orthogonal basis of regular harmonics

$$B_2 = \{z_1^2, 2z_1z_2, z_2^2, z_1z_2 - \frac{1}{2}z_2^2 j, |z_1|^2 - |z_2|^2, z_2z_1 + \frac{1}{2}z_2^2 j\}$$

of the six-dimensional right $\mathbb{H}$-module $U_2^\circ$.

(ii) The case $k = 3$. From the orthogonal bases $B_{3,0} = \{z_1^3, 3z_1^2 z_2, 3z_1z_2^2, z_2^3\}$, $B_{2,1} = \{z_1^2 z_2, 2z_1|z_2|^2 - z_1^2|z_1|^2, 2z_2|z_1|^2 - z_2^2|z_2|^2, z_2^2 z_1\}$, $B_{1,2} = \{h_1 = z_1 z_2^2, M(h_1) = -z_2 z_1^2, h_2 = 2z_2|z_2|^2 + z_2^2, M(h_2) = -2z_1|z_2|^2 + z_1^2|z_1|^2\}$, we get the orthogonal basis of regular harmonics

$$B_3 = \{z_1^3, 3z_1^2 z_2, 3z_1z_2^2, z_2^3, z_1^2 z_2 - \frac{1}{3}z_2^3 j, 2z_1|z_2|^2 - z_1^2|z_1|^2 - z_1^2 z_2 j, 2z_2^2|z_1|^2 - z_2^2|z_2|^2 + z_1^2 z_2^2 j\}$$

of the ten-dimensional right $\mathbb{H}$-module $U_3^\circ$.

In general, for any $k$, an orthogonal basis of $\mathcal{H}_{p,q}$ ($p + q = k$) is given by the polynomials $\{P_k^{\alpha}\}_{\alpha = 0, ..., k}$ defined by formula (6.14) in Sudbery [8]. The basis of $U_k$ obtained from these bases by means of Theorem 3 and applying the reflection $\gamma$ is essentially the same given in Proposition 8 of Sudbery [8].

Another spanning set of the space $\mathcal{H}_{p,q}$ is given by the functions

$$g_k^{p,q}(z_1, z_2) = (z_1 + \alpha z_2)^p(z_2 - \alpha z_1)^q \quad (\alpha \in \mathbb{C})$$

(cf. Rudin [7], §12.5.1). Since $M(g_k^{p,q}) = \frac{-1}{p+1}g_k^{p+1,q-1,\alpha}$ for $\alpha \neq 0$ and $M(g_k^{p,q}) = -\frac{q}{p+1}z_2^{2-1}z_2^{p+1}$, where we set $g_k^{p,q} \equiv 0$ if $p < 0$, from Theorem 3 we get that $U_k^\circ$ is spanned over $\mathbb{H}$ by the polynomials

$$R(g_k^{p,q}) = \begin{cases} g_k^{p,q} + \frac{(-1)^p q}{p+1} z_1^{p-1}z_2^{q-1,\alpha} \quad &\text{for } \alpha \neq 0 \\ \frac{-q}{p+1}z_1^{p-1}z_2^{q-1,\alpha} \quad &\text{for } \alpha = 0 \end{cases} \quad (\alpha \in \mathbb{C}, p + q = k)$$
Any choice of $k + 1$ distinct numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$ gives rise to a basis of $U_k^\psi$.

The results obtained in this paper enabled the writing of a Mathematica package [6], named RegularHarmonics, which implements efficient computations with regular and $\psi$-regular functions and with harmonic and holomorphic functions of two complex variables.

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