Quaternionic regular functions and the $\overline{\partial}$–Neumann problem in $\mathbb{C}^2$

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1. Some notations

\[ \mathbb{C}^2 \ni (z_1, z_2) = (x_0 + ix_1, x_2 + ix_3) \leftrightarrow \]
\[ q = z_1 + z_2j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H} \]

Let \( \Omega \) be a bounded domain in \( \mathbb{H} \approx \mathbb{C}^2 \). A quaternionic function \( f = f_1 + f_2j \in C^1(\Omega) \) is (left) regular on \( \Omega \) if

\[ Df = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0 \text{ on } \Omega, \]

\( f \) is (left) \( \psi \)-regular on \( \Omega \) if

\[ D'f = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} - k \frac{\partial f}{\partial x_3} = 0 \text{ on } \Omega. \]

Remarks 1. \( f \) is \( \psi \)-regular \( \Leftrightarrow \)

\[ \frac{\partial f_1}{\partial \overline{z}_1} = \overline{\frac{\partial f_2}{\partial z_2}}, \quad \frac{\partial f_1}{\partial \overline{z}_2} = -\overline{\frac{\partial f_2}{\partial z_1}} \Leftrightarrow \]

\[ \ast \partial f_1 = -\frac{1}{2} \partial(f_2 d\overline{z}_1 \wedge d\overline{z}_2). \]
2. Every regular or $\psi$-regular function is harmonic.
3. Every holomorphic map $(f_1, f_2)$ on $\Omega$ defines a $\psi$-regular function $f = f_1 + f_2j$.
4. If $\Omega$ is pseudoconvex, every complex harmonic function $f_1$ is the complex component of a $\psi$-regular function $f$ on $\Omega$.

2. Main results

2.1 A differential criterion for regularity

**Theorem 1.** $f = f_1 + f_2j \in C^1(\overline{\Omega})$ is $\psi$-regular on $\Omega$ if and only if $f$ is harmonic on $\Omega$ and

$$ (\overline{\partial}_n - jL)f = 0 \quad \text{on } \partial\Omega. \quad (*) $$

$\overline{\partial}_n f$ is the normal component of $\overline{\partial} f$ on $\partial\Omega$, defined by: $\overline{\partial}_n f d\sigma = *\overline{\partial} f|_{\partial\Omega}$,
$L$ is the tangential Cauchy-Riemann operator

$$L = \frac{1}{|\partial \rho|} \left( \frac{\partial \rho}{\partial \bar{z}_2 \bar{z}_1} \frac{\partial}{\partial \bar{z}_1} - \frac{\partial \rho}{\partial \bar{z}_1 \bar{z}_2} \frac{\partial}{\partial \bar{z}_2} \right).$$

**Remark.** Condition (*) generalizes both the CR-tangential equation $L(f) = 0$ and the condition $\bar{\partial}_n f = 0$ on $\partial \Omega$ that distinguishes holomorphic functions among complex harmonic functions (Aronov and Kytmanov).

The single equation (*) is equivalent to the following system of complex equations on $\partial \Omega$:

$$\bar{\partial}_n f_1 = -\overline{L(f_2)} \quad (C_1)$$
$$\bar{\partial}_n f_2 = \overline{L(f_1)} \quad (C_2)$$

A weak version of Theorem 1 gives a trace theorem:

**Theorem 2.** A continuous function $f : \partial \Omega \to \mathbb{H}$ is the trace of a $\psi$-regular function on $\Omega$ if and only if it satisfies the integral condition

$$\int_{\partial \Omega} \bar{f} \left( \bar{\partial}_n - jL \right) \phi \, d\sigma = 0 \quad \forall \phi \in \text{Harm}^1(\overline{\Omega}).$$
From Theorem 1 we immediately get the following result about regular functions:

**Theorem 3.** \( f = f_1 + f_2 j \in C^1(\Omega) \) is regular on \( \Omega \) if and only if \( f \) is harmonic on \( \Omega \) and

\[
(N - jT)f = 0 \text{ on } \partial \Omega, \quad \text{where}
\]
\[
N = \frac{\partial \rho}{\partial z_1} \frac{\partial}{\partial \bar{z}_1} + \frac{\partial \rho}{\partial z_2} \frac{\partial}{\partial \bar{z}_2}, \quad T = \frac{\partial \rho}{\partial z_2} \frac{\partial}{\partial \bar{z}_1} - \frac{\partial \rho}{\partial z_1} \frac{\partial}{\partial \bar{z}_2}. \]

2.2 A criterion for holomorphicity

When \( \partial \Omega \) is connected, Hartogs Theorem can be applied to improve the previous results. Now conditions

\[
\overline{\partial}_n f_1 = -\overline{L(f_2)} \quad \text{(C}_1\text{)}
\]
\[
\overline{\partial}_n f_2 = \overline{L(f_1)} \quad \text{(C}_2\text{)}
\]

are equivalent: one of them implies the \( \psi \)-regularity of \( f \).

**Remark.** The connectedness of \( \partial \Omega \) is a necessary assumption: consider a locally constant function on \( \partial \Omega \).
The equivalence of $C_1$ and $C_2$ can be used to get the following criterion for holomorphicity:

**Theorem 4.** Let $\Omega \subseteq \mathbb{C}^2$ be bounded, with connected boundary $\partial \Omega$. Let $a \in \mathbb{C}$. If $h \in C^1(\overline{\Omega})$ is complex harmonic and satisfies the condition $\overline{\partial}_n h = aL(h)$ on $\partial \Omega$, then $h$ is holomorphic on $\Omega$.

**Remark.** The case $a = 0$ is a theorem of Aronov and Kytmanov. Mixed differential conditions of this type have been studied in particular by Chirka and Kytmanov.
2.3 Regularity and the $\partial$-Neumann problem

The $\partial$-Neumann for complex functions can be formulated in the following way:

$$\partial_n g = \phi \text{ on } \partial \Omega, \quad g \text{ harmonic in } \Omega,$$

with compatibility condition

$$\int_{\partial \Omega} \phi \overline{h} d\sigma = 0 \quad \forall h \in \mathcal{O}(\overline{\Omega}).$$

If $\partial \Omega$ is connected and $C^\infty$–smooth and $\Omega$ is strongly pseudoconvex or weakly pseudoconvex with real analytic boundary, the solvability of $\partial$-Neumann problem (Kytmanov) applied to the equation

$$\partial_n f_2 = L(f_1) \quad (C_2)$$

allows to achieve the following:

**Theorem 5.** Let $f_1 : \partial \Omega \to \mathbb{C}$ be of class $C^\infty$. Then $f_1$ is the trace on $\partial \Omega$ of one complex component of a $\psi$-regular function $f$ on $\Omega$, of class $C^\infty$ on $\overline{\Omega}$. \quad \square
Remark. $f_2$ is determined up to a holomorphic function, so $f$ is uniquely determined by the orthogonality condition
\[
\int_{\partial \Omega} (f - f_1) \overline{h} d\sigma = 0 \quad \forall h \in \mathcal{O}(\overline{\Omega}).
\]
This defines a $\mathbb{C}$-linear operator
\[
R : C^\infty(\partial \Omega) \to M^\infty(\Omega).
\]

**Corollary 1.** Let $M^\infty(\Omega)$ be the right $\mathbb{H}$-module of left $\psi$-regular functions of class $C^\infty$ on $\overline{\Omega}$. The mapping $C$ defined by $C(f) = f_1|_{\partial \Omega}$ for every $f = f_1 + f_2 j \in M^\infty(\Omega)$ induces an isomorphism of real spaces
\[
\frac{M^\infty(\Omega)}{A^\infty(\Omega, \mathbb{C}^2)} \cong \frac{C^\infty(\partial \Omega)}{CR(\partial \Omega)}.
\]
2.4 An application: a product in $M^\infty(\Omega)$

The existence of a right inverse for $C$

$$M^\infty(\Omega) \xrightarrow{R/C} C^\infty(\partial\Omega) \iff C \circ R = Id_{C^\infty(\partial\Omega)}$$

allows to define a product in $M^\infty(\Omega)$, with respect to which $M^\infty(\Omega)$ becomes a commutative $\mathbb{R}$-algebra, with unity the constant function 1, and which contains $A^\infty(\Omega, \mathbb{C}^2)$ as a subalgebra with respect to the product

$$(f_1, f_2) \cdot (g_1, g_2) = (f_1g_1 + f_2g_2, f_1g_2 + f_2g_1).$$

Given $f, g \in M^\infty(\Omega)$, let

$$f \ast g = R(f_1g_1) - (f - R(f_1))j(g - R(g_1))$$

where $f_1 = C(f), g_1 = C(g)$.

Let $\phi : M^\infty(\Omega) \to M^\infty(\Omega)$

$$\phi(f) = f(1 + j).$$

The product $m_\Omega(f, g)$ can be defined as

$$m_\Omega(f, g) = \phi^{-1}(\phi(f) \ast \phi(g)).$$
3. The case of the unit ball

When $\Omega = B$ is the unit ball in $\mathbb{C}^2$, $S$ the unit sphere, the operators

$$\overline{\partial}_n = \overline{z}_1 \frac{\partial}{\partial \overline{z}_1} + \overline{z}_2 \frac{\partial}{\partial \overline{z}_2}, \quad L = z_2 \frac{\partial}{\partial \overline{z}_1} - z_1 \frac{\partial}{\partial \overline{z}_2}$$

preserve harmonicity. Condition (*) in Theorem 1 can be reformulated for polynomials. Let

$$D_k = \sum_{0 \leq l \leq k/2 - 1} \frac{(k - 2l - 1)!(2l - 1)!!}{k!(l + 1)!} 2^l \Delta^{l+1}.$$

**Theorem 6.** The restriction to $S$ of a homogeneous polynomial $f = f_1 + f_2 j$ of degree $k$ extends as a $\psi$-regular function into $B$ if and only if

$$(\overline{\partial}_n - D_k)f_1 + L(f_2) = 0 \quad \text{on } S.$$

It extends as a regular function if and only if

$$(N - D_k)f_1 + T(f_2) = 0 \quad \text{on } S.$$
Theorem 5 has the following homogeneous version:

**Theorem 7.** a) For every \( f_1 \in \mathcal{P}_k \) (complex \( k \)-homogeneous polynomial), there exists \( f_2 \in \mathcal{P}_k \) such that the trace of \( f = f_1 + f_2j \) on \( S \) extends as a \( \psi \)-regular polynomial of degree \( \leq k \) on \( \mathbb{H} \).

b) If \( f_1 \) is harmonic, then \( f \) belongs to the right \( \mathbb{H} \)-module \( U^\psi_k \) of \( \psi \)-regular homogeneous polynomials of degree \( k \).

The right inverse

\[
R : \mathcal{H}_k(S) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(S) \rightarrow U^\psi_k
\]

of \( C \) (\( \mathcal{H}_{p,q} \) the space of harmonic homogeneous polynomials of degree \( p \) in \( z \) and \( q \) in \( \bar{z} \), \( \mathcal{H}_k(S) \) the space of spherical harmonics) gives the following:

**Corollary 2.** The restriction first-component operator \( C \) induces isomorphisms

\[
\frac{U^\psi_k}{\mathcal{H}_{k,0} + \mathcal{H}_{k,0}j} \cong \frac{\mathcal{H}_k(S)}{\mathcal{H}_{k,0}(S)}.
\]
These isomorphisms can be applied to obtain $\mathbb{H}$-bases for $U_k^\psi$ starting from $\mathbb{C}$-bases of $\mathcal{H}_{p,q}$ ($p + q = k$). This construction preserves orthogonality w.r.t. $L^2(S)$.

Given bases $\{P_l\}$ of $\mathcal{H}_{p,q}$, a suitably chosen subset of the images

$$R(P_l) = \begin{cases} P_l & \text{if } q = 0 \\ P_l + \frac{1}{p + 1}L(P_l)j & \text{if } q > 0 \end{cases}$$

gives a $\mathbb{H}$-basis for $U_k^\psi$ ($\dim_{\mathbb{H}} U_k^\psi = \frac{(k+1)(k+2)}{2}$).

A possible choice for a $L^2(S)$-orthogonal basis of $\mathcal{H}_{p,q}$ is given by the $p+q+1$ polynomials

$$P_l(z_1, z_2) = \sum_{r = \max\{0, l-p\}}^{\min\{q, l\}} c_{l,r} z_1^{p-l+r} z_2^{l-r} \bar{z}_1^r \bar{z}_2^{q-r}$$

where $c_{l,r} = (-1)^r \binom{p}{l-r} \binom{q}{r}$ and $l = 0, \ldots, p+q$.

Cf. RegularHarmonics: a Mathematica 4.2 package available at

www.science.unitn.it
/~/perotti/regular harmonics.htm
4. Sketch of proofs

4.1 Theorem 1 (criterion for $\psi$-regularity)

The main point is a property of the differential form associated to the Cauchy-Fueter kernel for $\psi$-regular functions: its first complex component is the Bochner-Martinelli kernel in dimension 2 (Fueter–Vasilevski–Shapiro).

We show that the Bochner-Martinelli integral representation formula for harmonic functions, under condition (*), is the same as the Cauchy-Fueter integral representation formula, from which regularity follows.

4.2 Theorem 2 (trace theorem)

The result follows from the jump formula for the Cauchy-Fueter integral. Using again the property above, we show that the Cauchy-Fueter integral of $f \in C'(\partial \Omega)$ vanishes on the complement $\mathbb{C}^2 \setminus \overline{\Omega}$ under condition

$$\int_{\partial \Omega} \bar{f} \left( \overline{\partial_n} - jL \right) \phi \ d\sigma = 0 \ \forall \ \phi \in \text{Harm}^1(\overline{\Omega}).$$
When $\partial \Omega$ is connected and one of conditions $C_1$, $C_2$ (say $C_2$) holds, the Cauchy-Fueter integral of $f$ defines on $\mathbb{C}^2 \setminus \overline{\Omega}$ a complex valued $\psi$-regular function $F^- \Rightarrow$ a holomorphic function on $\mathbb{C}^2 \setminus \overline{\Omega} \Rightarrow$ a holomorphic function $\tilde{F}^-$ on $\mathbb{C}^2$.

In this way we get a $\psi$-regular function $F = F^+ - \tilde{F}^-|_{\overline{\Omega}}$ on $\Omega$, whose trace on $\partial \Omega$ is $f$.

4.3 Theorem 4 (criterion for holomorphicity)

Given $f = ah + hj$, condition $C_2$ is satisfied, and then $f$ is $\psi$-regular. From $\psi$-regularity equations we obtain

$$\overline{\partial} h = 0.$$  

4.4 Theorem 5 ($\overline{\partial}$-Neumann problem)

The result follows easily since $\phi = \overline{L(f_1)}$ satisfies the compatibility condition for $\overline{\partial}$-Neumann problem. Then there exists $f_2$ such that $\overline{\partial}_n f_2 = \overline{L(f_1)} \Rightarrow$ condition $C_2$ holds.
4.5 The case of the unit ball

For Theorem 6 we use a computation made by Kytmanov, who proved the analogous result for holomorphic extensions of homogeneous polynomials.

For Theorem 7, we suppose $f_1 \in \mathcal{H}_{p,q}$ and use Gauss formula for the harmonic extension into $B$ of the trace $f_1|_S$:

$$\tilde{f}_1 = \sum_{s \geq 0} g_{p-s,q-s},$$

where $g_{p-s,q-s}$ is the homogeneous harmonic polynomial of degree $p + q - 2s$ defined by

$$g_{p-s,q-s} = c_{p,q,s} \sum_{j \geq 0} \frac{(-1)^j(p+q-j-2s)!}{j!} |z|^{2j} \Delta^j z^{j+s} f_1.$$

The equation $\overline{\partial}_n f_2 = \overline{L}(f_1)$ can now be solved easily since

$$\overline{\partial}_n \overline{L}(g_{p-s,q-s}) = (p - s + 1) \overline{L}(g_{p-s,q-s}).$$
4.6 Bases of $U_k^\psi$

Let $\mathcal{B}_{p,q}$ denote a complex base of the space $\mathcal{H}_{p,q}(S)$ ($p + q = k$). Then:

(i) if $k = 2m$ is even, a basis of $U_k^\psi$ over $\mathbb{H}$ is given by the set

$$\mathcal{B}_k = \{ R(h) : h \in \mathcal{B}_{p,q}, p+q = k, 0 \leq q \leq p \leq k \}. $$

(ii) if $k = 2m + 1$ is odd, a basis of $U_k^\psi$ over $\mathbb{H}$ is given by

$$\mathcal{B}_k = \{ R(h) : h \in \mathcal{B}_{p,q}, p+q = k, 0 \leq q < p \leq k \} 
\cup \{ R(h_1), \ldots, R(h_{m+1}) \}, $$

where $h_1, \ldots, h_{m+1}$ are chosen such that the set

$$\left\{ h_1, \frac{1}{p+1}L(h_1), \ldots, h_{m+1}, \frac{1}{p+1}L(h_{m+1}) \right\}$$

forms a complex basis of $\mathcal{H}_{m,m+1}(S)$. 
4.7 The product in $M^\infty(B)$

On the unit ball we have explicit formulas for harmonic continuation of polynomials and for the operator $R$.

**Example.** The product of the $\psi$-regular, not holomorphic function

$$ f = (\bar{z}_1 + \bar{z}_2) + (\bar{z}_2 - \bar{z}_1)j $$

with itself is the $\psi$-regular function

$$ m_B(f, f) = (2\bar{z}_1^2 + 4z_1\bar{z}_2) + (4z_1\bar{z}_2 - 2\bar{z}_1^2)j $$

and the product of $f$ and $g = z_1 - z_1j$ is

$$ m_B(f, g) = m_B(g, f) = (\|z_1\|^2 - \|z_2\|^2 + \bar{z}_1\bar{z}_2 + 1) + (\|z_2\|^2 - \|z_1\|^2 + \bar{z}_1\bar{z}_2 - 1)j. $$