RegularHarmonics
a Mathematica 4.2 package
for computing with regular
quaternionic functions

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Trento, April 2004
RegularHarmonics

by Alessandro Perotti
Version 1.2 - April 2004

This package implements computations with Fueter-regular quaternionic functions and harmonic functions of two complex variables.

■ Reference

■ Title

RegularHarmonics - Version 1.2 - April 2004

■ Author

Alessandro Perotti

■ Summary

This package implements computations with Fueter-regular quaternionic functions and harmonic functions of two complex variables.

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■ Mathematica Version: 4.2

■ References

- http://www.science.unitn.it/~perotti/RegularHarmonics.htm
■ Interface

■ Initial messages and package context

Off[General::"spell"];Off[General::"spell1"]
Print["RegularHarmonics by A.Perotti, Version 1.2, April 2004"]
Print["This package implements computations with Fueter-regular quaternionic polynomials and harmonic functions of two complex variables."]
Print["Additional information are available on the world wide web at the page http://www.science.unitn.it/~perotti/RegularHarmonics.htm"]
Print["Send comments and bug reports to: perotti@science.unitn.it"]

BeginPackage["RegularHarmonics`"]

■ Usage messages

■ Error messages

RegularHarmonics::notpoly = "`1` is not a polynomial in `2`"

■ Implementation

■ Begin the private context

Begin["Private""]

■ Unprotect system functions

protected = Unprotect[Conjugate,D]

■ Definition of auxiliary functions

■ Norms

■ ComplexNorm

ComplexNorm[a_?NumericQ*z_]:=Abs[a] ComplexNorm[z]
ComplexNorm[Conjugate[z_]]:=ComplexNorm[z]
ComplexNorm[a_?NumericQ]:=Abs[a]
Format[ComplexNorm[z_],StandardForm]:=BracketingBar[z]
symbolC

\[
\text{symbolC}[f\_,\ z\_\text{Symbol}\_z] \:= \text{Module}[],\ \text{Format}[z[i\_],\ \text{StandardForm}] = z_i;
\]

Symbol[SymbolName[z <> "-"][i\_] := Conjugate[z[i\_]];

ReplaceAll[f, \{z[i\_] \to z[i], \text{Subscript}[z, i\_] \to z[i]\} /. 

ComplexNorm[z] \to \text{Sqrt}[z[1]\ Conjugate[z[1]] + z[2]\ Conjugate[z[2]]] /. 

ComplexNorm[z[i\_]] \to \text{Sqrt}[z[i]\ Conjugate[z[i]]]]

ToComplexNorm

\[
\text{ToComplexNorm}[f\_,\ z\_\text{Symbol}\_z] \:= \text{Module}[\{\text{cn}\},\ 
\text{Simplify}[\text{symbolC}[f, z], z[1]\ Conjugate[z[1]] + z[2]\ Conjugate[z[2]] = c] /. 
\text{cn} \to \text{ComplexNorm}[z]^2 / z[i\_]^<(\text{n}\_\text{Integer}:1)\ Conjugate[z[i\_]]^<(\text{m}\_\text{Integer}:1) \to 
\text{ComplexNorm}[z[i]]^<(2\text{Min}[\text{n},\ \text{m}])\ z[i]^<(\text{n} - \text{Min}[\text{n},\ \text{m}])
\]

\text{Conjugate}[z[i]]^<(\text{m} - \text{Min}[\text{n},\ \text{m}])
\]

\text{SetAttributes}[\text{ToComplexNorm},\ \text{Listable}]

ToRealNorm

\[
\text{ToRealNorm}[f\_,\ x\_\text{Symbol}\_x] \:= \text{Module}[\{\text{rn}\},\ 
\text{Simplify}[f / x[i\_] \to x[i], \text{Sum}[x[i]^2, \{i, 0, 3\}] = \text{rn}] /. \text{rn} \to \text{ComplexNorm}[x]^2
\]

\text{SetAttributes}[\text{ToRealNorm},\ \text{Listable}]

Tonorm

Tonorm = False;

\text{ToCxNorm}[f\_,\ z\_\text{Symbol}\_z] := \text{If}[\text{Tonorm},\ \text{ToComplexNorm}[f, z], f, f]

Auxiliary functions for polynomials

NormalSeries

\[
\text{NormalSeries}[f\_,\ \text{n}\_\text{Integer},\ z\_\text{Symbol}\_z] \:= \text{Module}[\{\text{zb},\ \text{t}\},\ 
\text{Normal}[\text{Series}[\text{symbolC}[f, z] / . \text{Conjugate}[z[i\_]] \to \text{t}\ \text{zb}[i] / . \text{z[i\_]} \to \text{t}\ \text{z}[i], \{\text{t}, 0, \text{n}\}] /. \text{t} \to 1 /. \text{zb[i\_]} \to \text{Conjugate}[z[i]]]
\]
HomogeneousParts

\begin{verbatim}
HomogeneousParts[f__, z_Symbol: z] := Module[{e, t, l, k, st},
  e = symbolC[f, z];
  If[PolynomialQ[e, {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}],
    t = First[Internal`DistributedTermsList[
      e, {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}]];
    While[Length[t] == 0, t = {((0, 0, 0, 0))}; l = {};
    While[Length[t] > 0,
      k = Plus @@ First[t][[1]];
      st = Select[t, Plus @@ First[#[k]] == k &]; t = Complement[t, st];
      l = Append[l, {Internal`FromDistributedTermsList[
        {st, {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}}, k]]];
    Sort[1, OrderedQ[{#1[2], #2[2]}] &], Message[
      RegularHarmonics::notpoly, f, z]]]
  HomogeneousParts[f__, n_Integer, z_Symbol: z] :=
  HomogeneousParts[NormalSeries[f, n, z, z]]
  SetAttributes[HomogeneousParts, Listable]
\end{verbatim}

ComplexHomogeneousParts

\begin{verbatim}
ComplexHomogeneousParts[f__, z_Symbol: z] := Module[{e, t, l, k, st},
  e = symbolC[f, z];
  If[PolynomialQ[e, {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}],
    t = First[Internal`DistributedTermsList[e,
      {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}]]]; l = {};
    While[Length[t] > 0,
      p = Plus @@ First[t][{1, {2}}]; q = Plus @@ First[t][{1, {3, 4}}];
      st = Select[Select[t, Plus @@ #[[1, 2]] &],
        Plus @@ #[[1, {3, 4}]] &]; t = Complement[t, st];
      l = Append[l, {Internal`FromDistributedTermsList[
        {st, {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}}, {p, q}]}];
    Sort[Sort[1, OrderedQ[{Plus @@ #1[[2]], Plus @@ #2[[2]]}]] &], Message[
      RegularHarmonics::notpoly, f, z]]]
  ComplexHomogeneousParts[f__, n_Integer, z_Symbol: z] :=
  ComplexHomogeneousParts[NormalSeries[f, n, z, z]]
  SetAttributes[ComplexHomogeneousParts, Listable]
\end{verbatim}

TotalDegree

\begin{verbatim}
TotalDegree[f__, z_Symbol: z] :=
\end{verbatim}

LeadingTerm

\begin{verbatim}
LeadingTerm[f__, z_Symbol: z] :=
  Module[{l = First[Internal`DistributedTermsList[symbolC[f, z],
      {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}]]},
    l = Sort[Sort[1, OrderedQ[{Plus @@ #1[[2]], Plus @@ #2[[2]]}]] &],
    Internal`FromDistributedTermsList[
      {Last[l], {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}}]]
    SetAttributes[LeadingTerm, Listable]
\end{verbatim}
- Miscellaneous

- OnS (trace on the boundary of the unit ball)

\[ \text{OnS}[f, z] := \text{Simplify}[[f \cdot \text{Conjugate}[z]] \rightarrow z, z[1] \text{Conjugate}[z[1]] + z[2] \text{Conjugate}[z[2]] = 1] \]

- **Definition of principal functions**

- Laplacian and Kelvin Transform

- Laplacian (real)

\[ \text{Laplace}[f, x] := \text{Simplify}[[\text{Sum}[[D[f] \rightarrow x[j], j, 0, 3]], 2] \rightarrow x[j]] \]

- ComplexLaplacian

\[ \text{ComplexLaplace}[f, z] := \text{Module}[[e], e = \text{Expand}[[\text{SymbolC}[f, z]; \text{Simplify}[[\text{Sum}[[D[e, \text{Conjugate}[z[j]], z[j]], j, 2]]]] \]

- KelvinTransform

\[ \text{KelvinTransform}[f, z] := \text{Module}[[h], h = \text{HomogeneousParts}[[\text{SymbolC}[f, z]]; \text{Sum}[[h[i, 1]] \text{ComplexNorm}[z]^{(-2 - 2 h[i, 2])}, i, \text{Length}[h]]] \]

- Field Conversions

- RtoC

\[ \text{RtoC}[[f_1, f_2], x, z] := \text{Module}[[x, 2 \rightarrow x, x[j] \rightarrow x[j] - x[i] \cdot \text{Im} x[i]; \text{ComplexExpand}[[\text{Re}[[e]], \text{Im}[[e]]]; x[j] \rightarrow x[j]] \]

- CtoR

\[ \text{CtoR}[[f, x], z] := \text{Module}[[x, 2 \rightarrow x, x[i] \rightarrow x[i] - x[i - 1] \cdot \text{Im} x[i - 1]; \text{ComplexExpand}[[\text{Re}[[e]], \text{Im}[[e]]]; x[j] \rightarrow x[j]] \]
\[ CtoH[f_\text{List}, z_{\text{Symbol}}: z, x_{\text{Symbol}}: x] := \]
\[ \text{Flatten[Table[CtoR[f, z, x]], \{All, i\}, \{i, 2\}]]; \]

\[ HtoC[f\text{0}_\_, f\text{1}_\_, f\text{2}_\_, f\text{3}_\_], x_{\text{Symbol}}: x, z_{\text{Symbol}}: z] := \]
\[ \{\text{RtoC}[f\text{0}, f\text{1}], x, z], \text{RtoC}[[f\text{2}, f\text{3}], x, z]\}; \]

\section*{Cauchy-Riemann-Fueter equations for regular and \(\psi\)-regular functions and related boundary operators}

\[ \text{CRF}[\{f\text{1}_\_, f\text{2}_\_\}, z_{\text{Symbol}}: z] := \]
\[ \text{Module}[\{e1, e2\}, \{e1, e2\} = \text{Expand[symbolC}[[f1, \text{Conjugate}[f2]], z]]; \]
\[ \text{Simplify}[[\text{D}[f1, \text{Conjugate}[z[1]]] - \text{D}[\text{Conjugate}[f2], \text{Conjugate}[z[2]]], \]
\[ \text{D}[f1, z[2]] + \text{D}[\text{Conjugate}[f2], z[1]]]]; \]

\[ \text{PsiCRF}[\{f\text{1}_\_, f\text{2}_\_\}, z_{\text{Symbol}}: z] := \]
\[ \text{Module}[\{e1, e2\}, \{e1, e2\} = \text{Expand[symbolC}[[f1, \text{Conjugate}[f2]], z]]; \text{Simplify}[
\[ \text{D}[e1, \text{Conjugate}[z[1]]] - \text{D}[e2, z[2]], \text{D}[e1, \text{Conjugate}[z[2]]] + \text{D}[e2, z[1]]]]] \]

\[ \text{DbarN}[f\_\_, z_{\text{Symbol}}: z] := \]
\[ \text{Module}[\{e, zb\}, e = \text{Expand[symbolC}[f, z] /. \text{Conjugate}[z[i\_]] \rightarrow zb[i]]; \]
\[ \text{Sum}[zb[i] \text{D}[e, zb[i]], \{i, 2\}] /. \text{zb}[i\_] \rightarrow \text{Conjugate}[z[i]]] \]

\[ \text{L}[f\_\_, z_{\text{Symbol}}: z] := \text{Module}[\{zp, e\}, \text{zp}[i\_\?\text{OddQ}]=z[i+1]; \]
\[ \text{zp}[i\_\?\text{EvenQ}]=-z[i-1]; \text{symbolC}[0, z]; \text{Sum}[\text{zp}[i] \text{D}[f, \text{Conjugate}[z[i]]], \{i, 2\}] \]

\[ \text{Lbar}[f\_\_, z_{\text{Symbol}}: z] := \text{Module}[\{zp, zb, e\}, \]
\[ \text{zp}[i\_\?\text{OddQ}]=\text{Conjugate}[z[i+1]]; \text{zp}[i\_\?\text{EvenQ}]=-\text{Conjugate}[z[i-1]]; \]
\[ e = \text{Expand[symbolC}[f, z] /. \text{Conjugate}[z[i\_]] \rightarrow zb[i]]; \]
\[ \text{Sum}[\text{zp}[i] \text{D}[e, z[i]], \{i, 2\}] /. \text{zb}[i\_] \rightarrow \text{Conjugate}[z[i]]] \]
- NFueter

```
NFueter[f_, z_Symbol : z] :=
Module[{e, zb}, e = Expand[SymbolC[f, z] /. Conjugate[z[i__]] -> zb[i]];
```

- TFueter

```
TFueter[f_, z_Symbol : z] :=
Module[{zb, e}, e = Expand[SymbolC[f, z] /. Conjugate[z[i__]] -> zb[i]];
```

- PsiRegularQ

```
PsiRegularQ[{f1_, f2_}, z_Symbol : z] :=
{OnS[DbarN[f1, z] + Lbar[Conjugate[f2], z], z], ComplexLaplacian[{f1, f2}, z]} ===
{0, (0, 0)}
```

- RegularQ

```
RegularQ[{f1_, f2_}, z_Symbol : z] :=
{OnS[NFueter[f1, z] \[Conjugate][TFueter[f2, z]], z],
  ComplexLaplacian[{f1, f2}, z]} === {0, (0, 0)}
```

- Gauss formulas for harmonic extension and harmonic decomposition of polynomials on the unit ball and on the exterior of the unit ball

- GaussExtension

```
HomGaussExtension[e_, k_Integer, z_Symbol : z] := Module[{f, k2, lap},
   f = SymbolC[e, z];
   k2 = Floor[k/2];
   lap[0] = f; lap[1] = ComplexLaplacian[f, z];
   Do[lap[j] = ComplexLaplacian[lap[j - 1], z], {j, 2, k2}];
   Sum[(k - 2 s + 1) / (s! (k - s + 1)!) Sum[(-1)^j (k - j - 2 s)! / j!
   Sum[z[i] Conjugate[z[i]], {i, 2}]^j lap[j + s], {j, 0, k2 - s}], {s, 0, k2}]]

GaussExtension[e_, z_Symbol : z] := Module[{hp, f = SymbolC[e, z]},
   If[PolynomialQ[f, {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}],
      hp = HomogeneousParts[f, z];
      ToCxNorm[Expand[Sum[HomGaussExtension[hp[[i, 1]], hp[[i, 2]], z],
          {i, Length[hp]}]], z], Message[RegularHarmonics::notpoly, f, z]]
   GaussExtension[e_, n_Integer, z_Symbol : z] := GaussExtension[NormalSeries[e, n, z], z]
   SetAttributes[GaussExtension, Listable]
```
GaussForm

HomGaussForm[e_, k_Integer, z_Symbol: z] := Module[{f, k2, lap},
  f = symbolC[e, z]; k2 = Floor[k/2];
  lap[0] = f; lap[1] = ComplexLaplacian[f, z];
  Do[lap[j] = ComplexLaplacian[lap[j-1], z], {j, 2, k2}];
  Table[
    Sum[(k-2 s + 1) / (s! (k-s-1)!) (-1)^j (k-j - 2 s) / j! Sum[z[i] Conjugate[z[i]], {i, 2}]^j lap[j+s], {j, 0, k2-s}] / (2 s), {s, 0, k2}]
GaussForm[e_, z_Symbol: z] := Module[{hp, ghp, st, l, k}, f = symbolC[e, z];
  If[PolynomialQ[f, {z[1], z[2]}, Conjugate[z[1]], Conjugate[z[2]]],
    hp = HomogeneousParts[f, z];
    ghp = Flatten[Table[HomGaussForm[hp[[i, 1]], hp[[i, 2]], z], {i, Length[hp]}]], 1];
  l = {1};
  While[Length[ghp] > 0, k = First[ghp][[2]]; st = Select[ghp, #[[2]] == k &]; ghp = Complement[ghp, st];
    l = Append[l, {Plus @@ st[[All, 1]], k}];]
  ToCxNorm[Expand[Sort[1, OrderedQ[#1[[2]], #2[[2]]] &]], z],
   Message[RegularHarmonics::notpoly, f, z]]
GaussForm[e_, n_Integer, z_Symbol: z] := GaussForm[NormalSeries[e, n, z, z]
  SetAttributes[GaussForm, Listable]

ExteriorGaussExtension

ExteriorHomGaussExtension[e_, k_Integer, z_Symbol: z] := Module[{f, k2, lap},
  f = symbolC[e, z]; k2 = Floor[k/2];
  lap[0] = f; lap[1] = ComplexLaplacian[f, z];
  Do[lap[j] = ComplexLaplacian[lap[j-1], z], {j, 2, k2}];
  Sum[(k-2 s + 1) / (s! (k-s-1)!) (-1)^j (k-j - 2 s) / j! Sum[z[i] Conjugate[z[i]], {i, 2}]^j lap[j+s], {j, 0, k2-s}] / (2 s), {s, 0, k2}]
ExteriorGaussExtension[e_, z_Symbol: z] := Module[{hp}, f = symbolC[e, z];
  If[PolynomialQ[f, {z[1], z[2]}, Conjugate[z[1]], Conjugate[z[2]]],
    hp = HomogeneousParts[f, z];
    ToCxNorm[Expand[Sum[ExteriorHomGaussExtension[hp[[i, 1]], hp[[i, 2]], z], {i, Length[hp]}]], z],
   Message[RegularHarmonics::notpoly, f, z]]
ExteriorGaussExtension[e_, n_Integer, z_Symbol: z] :=
  ExteriorGaussExtension[NormalSeries[e, n, z, z]
  SetAttributes[ExteriorGaussExtension, Listable]

Regular and \psi-regular extensions of polynomials on the unit ball

Dk

Dk[e_, k_, z_Symbol: z] := Module[{f, k2, lap},
  f = symbolC[e, z]; k2 = Floor[k/2];
  lap[0] = f; lap[1] = ComplexLaplacian[f, z];
  Do[lap[j] = ComplexLaplacian[lap[j-1], z], {j, 2, k2}];
  1 / k! Sum[2^l (k - 2 l - 1)! (2 l - 1)!! / (l+1)! lap[l+1], {l, 0, k2-1}]]
PsiRegularExtensionQ

PsiRegularExtensionQ[{f1_, f2_}, z_Symbol : z] :=
Module[{hp, hp = HomogeneousParts[f1, z]; OnS[DbarN[f1, z] + LbarConjugate[f2, z] -
Sum[Dk[hp[[i, 1]], hp[[i, 2]], z], {i, Length[hp]}], z] == 0}

RegularExtensionQ

RegularExtensionQ[{f1_, f2_}, z_Symbol : z] :=
Module[{hp, hp = HomogeneousParts[f1, z];
OnS[NFuter[f1, z] + Conjugate[TFuter[f2, z]] -
Sum[Dk[hp[[i, 1]], hp[[i, 2]], z], {i, Length[hp]}], z] == 0}

PsiRegularExtension

PsiRegularExtension[{f1_, f2_}, z_Symbol : z] :=
Module[{ge, ge = GaussExtension[{f1, f2}, z]; If[PsiRegularQ[ge, z], ge, "No ψ-regular extension"]}
PsiRegularExtensionQ[ge, z] :=
Module[{chp, ge, f2}, chp = ComplexHomogeneousParts[f1, z];
ge = Table[HomGaussForm[chp[[i, 1]], Plus@chp[[i, 2]], z], {i, Length[chp]}];
f2 = ToCxNorm[Expand[
Sum[Sum[1/(First[chp[[i, 2]]] - s + 1)] LbarConjugate[First[ge[[i, s + 1]]]], z],
{s, 0, Min[chp[[i, 2]]]], {i, Length[chp]}}, {z}; (GaussExtension[f1, z, f2]}
PsiRegularExtensionQ[f1_, n_Integer, z_Symbol : z] :=
PsiRegularExtension[NormalSeries[f1, n, z], z]

RegularExtension

RegularExtension[{f1_, f2_}, z_Symbol : z] := Module[{ge},
ge = GaussExtension[{f1, f2}, z]; If[RegularQ[ge, z], ge, "No regular extension"]}
RegularExtensionQ[f1_, z_Symbol : z] := Expand[
RegularExtensionQ[f1_, n_Integer, z_Symbol : z] :=
PsiRegularExtension[NormalSeries[f1, n, z], z]

Sphere and ball integrals

SphereIntegral[f_List, z_Symbol : z] :=
{SphereIntegral[f[[1]], z], SphereIntegral[f[[2]], z]}

SphereIntegral[f_, z_Symbol : z] := Module[{e, t, a},
e = symbolC[f, z];
If[PolynomialQ[e, {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}],
t = First[Internal`DistributedTermsList[e,
{z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}]];]
Sum[If[a = Part[t, i][{1, 1, 2}] == Part[t, i][{1, 3, 4}],
Part[t, i][2] Times @@ (a) / (Plus @@ a + 1)!, 0], {i, Length[t]}],
Message[RegularHarmonics::notpoly, f, z]]}
SphereProduct

SphereProduct[f1_, f2_, {g1_, g2_}, z_Symbol: z] := Module[{e1, h1, e2, h2},
{e1, h1, e2, h2} = symbolC[{f1, g1, f2, g2}, z];
SphereIntegral[
{h1 Conjugate[e1] + e2 Conjugate[h2], h2 Conjugate[e1] - e2 Conjugate[h1]}, z]]

SphereIntegral[f_, g_, z_Symbol: z] := Module[{f1, g1},
{f1, g1} = symbolC[{f, g}, z]; SphereIntegral[f1 Conjugate[g1], z]]

SphereNorm

SphereNorm[f1_, f2_, z_Symbol: z] := Module[{e1, e2},
{e1, e2} = symbolC[{f1, f2}, z];
Sqrt[SphereIntegral[e1 Conjugate[e1] + e2 Conjugate[e2], z]]

SphereNorm[f_, z_Symbol: z] := Module[{e},
e = symbolC[f, z];
Sqrt[SphereIntegral[e Conjugate[e], z]]

BallIntegral

BallIntegral[f_List, z_Symbol: z] :=
{BallIntegral[f[[1]], z], BallIntegral[f[[2]], z]}

BallIntegral[f_, z_Symbol: z] := Module[{e, t, a},
e = symbolC[f, z];
If[PolynomialQ[e, {z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]}],
t = First[Internal`DistributedTermsList[e, z[1], z[2], Conjugate[z[1]], Conjugate[z[2]]]]];
Sum[If[(a = Part[t, i][[1, {1, 2}]] == Part[t, i][[1, {3, 4}]]],
2 Part[t, i][[2]] Times @@ (a!) / (Plus @@ (a + 2) !), 0], i, Length[t]],
Message[RegularHarmonics::notpoly, f, z]]

BallProduct

BallProduct[f1_, f2_, {g1_, g2_}, z_Symbol: z] :=
Module[{e1, h1, e2, h2},
{e1, h1, e2, h2} = symbolC[{f1, g1, f2, g2}, z];
BallIntegral[
{h1 Conjugate[e1] + e2 Conjugate[h2], h2 Conjugate[e1] - e2 Conjugate[h1]}, z]]

BallProduct[f_, g_, z_Symbol: z] := Module[{f1, g1},
{f1, g1} = symbolC[{f, g}, z]; BallIntegral[f1 Conjugate[g1], z]]

BallNorm

BallNorm[f1_, f2_, z_Symbol: z] := Module[{e1, e2},
{e1, e2} = symbolC[{f1, f2}, z];
Sqrt[BallIntegral[e1 Conjugate[e1] + e2 Conjugate[e2], z]]

BallNorm[f_, z_Symbol: z] := Module[{e},
e = symbolC[f, z];
Sqrt[BallIntegral[e Conjugate[e], z]]]
## Spherical harmonics bases

### BasisP

\[
\text{CBasisP}[p_, \ q_, \ l_, \ r_] := (-1)^r \text{Binomial}[p, 1-r] \text{Binomial}[q, r] \\
\text{BasisP}[p\_\text{\_Integer}, q\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \text{Module}[[t, \text{symbolC}[0, z]]; \\
t = \text{Table}[\text{Sum}[\text{CBasisP}[p, q, 1, r] z[1]^((p-1+r) z[2]^((1-r) \text{Conjugate}[z[1]])^r \\
\text{Conjugate}[z[2]]^((q-r), \{r, \text{Max}[1-p, 0], \text{Min}[q, 1]\}, \{1, 0, \text{Floor}[(p+q)/2]\}); \\
\text{Join}[\text{If}[\text{EvenQ}[p+q], \text{Delete}[t, -1], t], \text{Reverse}[t]/\{z[1] \rightarrow z[2], z[2] \rightarrow z[1]\}]] \\
\text{BasisP}[k\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \text{Module}[[t, \\
t = \text{Table}[\text{BasisP}[k-q, q, z], \{q, 0, \text{Floor}[k/2]\}]; \text{Flatten}[
\text{Join}[\text{If}[\text{EvenQ}[k], \text{Delete}[t, -1], t], \text{Reverse}[t]/.z[_] \rightarrow \text{Conjugate}[z[i]]]]
\]

### RealBasisP

\[
\text{RealBasisP}[k\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \\
\text{Module}[[t, t = \text{Table}[\text{BasisP}[k-q, q, z], \{q, 0, \text{Floor}[k/2]\}]; t = \text{Join}[\text{Delete}[
\text{Flatten}[t + \text{Conjugate}[t]], \text{If}[\text{EvenQ}[k], \text{Table}[[i, \{i, -k/2, -1\}, \{\}\}], \text{Delete}[
\text{Flatten}[\text{t}(-t + \text{Conjugate}[t])], \text{If}[\text{EvenQ}[k], \text{Table}[[i, \{i, -k/2-1, -1\}, \{\}]]]; \\
\text{Expand}[\text{If}[\text{EvenQ}[k], \text{ReplacePart}[t, t[[k^2/2+k+1]]/2, k^2/2+k+1, t]]]
\]

### ONBasisP

\[
\text{ONBasisP}[p\_\text{\_Integer}, q\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \text{Module}[[c, t, \text{symbolC}[0, z]]; \\
\text{Do}[[c[1, r] = \text{CBasisP}[p, q, 1, r], \\
\{1, 0, \text{Floor}[(p+q)/2]\}, \{r, \text{Max}[0, 1-p], \text{Min}[q, 1]\}]; \\
t = \text{Table}[\text{Sum}[c[1, r] z[1]^((p-1+r) z[2]^((1-r) \text{Conjugate}[z[1]])^r \\
\text{Conjugate}[z[2]]^((q-r), \{r, \text{Max}[1-p, 0], \text{Min}[q, 1]\}] \\
\text{Sqrt}[(p+q+1)!]/\text{Sqrt}[\text{Sum}[c[1, s] \text{Sum}[c[1, r] (p-1+r+s) (q+1-r-s), \\
\{r, \text{Max}[0, 1-p], \text{Min}[q, 1]\}], \{s, \text{Max}[0, 1-p], \text{Min}[q, 1]\}]], \\
\{1, 0, \text{Floor}[(p+q)/2]\}]; \text{Join}[\text{If}[\text{EvenQ}[p+q], \text{Delete}[t, -1], t], \\
\text{Reverse}[t]/\{z[1] \rightarrow z[2], z[2] \rightarrow z[1]\}]] \\
\text{ONBasisP}[k\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \text{Module}[[t, \\
t = \text{Table}[\text{ONBasisP}[k-q, q, z], \{q, 0, \text{Floor}[k/2]\}]; \text{Flatten}[
\text{Join}[\text{If}[\text{EvenQ}[k], \text{Delete}[t, -1], t], \text{Reverse}[t]/.z[_] \rightarrow \text{Conjugate}[z[i]]]]
\]

### BallONBasisP

\[
\text{BallONBasisP}[p\_\text{\_Integer}, q\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \text{ONBasisP}[p, q, z] \text{Sqrt}[(p+q+2)/2] \\
\text{BallONBasisP}[k\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \text{ONBasisP}[k, z] \text{Sqrt}[(k+2)/2] \\
\text{RealBallONBasisP}[k\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \\
\text{Module}[[t, t = \text{Table}[\text{ONBasisP}[k-q, q, z]/\text{Sqrt}[2], \{q, 0, \text{Floor}[k/2]\}]; \\
t = \text{Join}[\text{Delete}[\text{Flatten}[t + \text{Conjugate}[t]], \\
\text{If}[\text{EvenQ}[k], \text{Table}[[i, \{i, -k/2, -1\}, \{\}\}], \text{Delete}[
\text{Flatten}[\text{t}(-t + \text{Conjugate}[t])], \text{If}[\text{EvenQ}[k], \text{Table}[[i, \{i, -k/2-1, -1\}, \{\}]]]; \\
\text{Expand}[\text{If}[\text{EvenQ}[k], \text{ReplacePart}[t, t[[k^2/2+k+1]]/\text{Sqrt}[2], k^2/2+k+1, t]]]
\]
\]

\[
\text{RealBallONBasisP}[k\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \text{RealONBasisP}[k, z] \text{Sqrt}[(k+2)/2] \\
\text{Basist}[p\_\text{\_Integer}, q\_\text{\_Integer}, z\_\text{\_Symbol}: z] := \\
(\text{symbolC}[0, z]; \{z[1] + \alpha z[2]\})^p (\text{Conjugate}[z[2]] - \alpha \text{Conjugate}[z[1]])^q
\]
- Regular and $\psi$-regular spherical harmonics bases

- PsiRegularBasis

$$\text{PsiRegularBasis}[k_?\text{EvenQ}, z_\text{Symbol}: z] :=$$
$$\text{Module}[(b), \text{Do}[b[p, k - p] = \text{BasisP}[p, k - p, z], \{p, k/2, k\}];$$
$$\text{Expand}[@\text{Flatten}[\text{Table}[[b[p, k - p][[i]], (p + 1)^(-1) \text{Conjugate}[L[b[p, k - p][[i]], z]], \{p, k, k/2, -1}, \{i, k + 1\}], 1]]]$$

- PsiRegularONBasis

$$\text{PsiRegularONBasis}[k_?\text{EvenQ}, z_\text{Symbol}: z] :=$$
$$\text{Module}[(b), \text{Do}[b[p, k - p] = \text{ONBasisP}[p, k - p, z], \{p, k/2, k\}]; \text{Expand}[\text{Delete}[$$
$$\text{Flatten}[\text{Table}[[b[p, k - p][[i]], (p + 1)^(-1) \text{Conjugate}[L[b[p, k - p][[i]], z]], \{p, k, (k - 1)/2, -1}, \{i, k + 1\}], 1]]]$$

- PsiRegularBallONBasis

$$\text{PsiRegularBallONBasis}[k_\text{Integer}, z_\text{Symbol}: z] :=$$
$$\text{PsiRegularONBasis}[k, z] \text{ Sqrt}[(k + 2)/2]$$

- RegularBasis

$$\text{RegularBasis}[k_\text{Integer}, z_\text{Symbol}: z] :=$$
$$\text{Expand}[\text{PsiRegularBasis}[k, z] / . \text{z}[2] \rightarrow \text{Conjugate}[\text{z}[2]]]$$

- RegularONBasis

$$\text{RegularONBasis}[k_\text{Integer}, z_\text{Symbol}: z] :=$$
$$\text{Expand}[\text{PsiRegularONBasis}[k, z] / . \text{z}[2] \rightarrow \text{Conjugate}[\text{z}[2]]]$$

- RegularBallONBasis

$$\text{RegularBallONBasis}[k_\text{Integer}, z_\text{Symbol}: z] :=$$
$$\text{Expand}[\text{PsiRegularBallONBasis}[k, z] / . \text{z}[2] \rightarrow \text{Conjugate}[\text{z}[2]]]$$
New definitions for system functions

Conjugate[z_Plus]:=Conjugate/@z
Conjugate[z_Times]:=Conjugate/@z
Conjugate[z_^n_Integer]:=Conjugate[z]^n
Conjugate[Conjugate[z_]]:=z
Conjugate'[z_]:=0
Format[Conjugate[z_],StandardForm]:=OverBar[z]

D[f_,Conjugate[z_]]:=Conjugate[D[Conjugate[f],z]]

(*MakeExpression[RowBox[{"OverBar","[",x_,"\"]"}],FullForm]:=MakeExpression[RowBox[{"Conjugate","[",x,"\"]"}],FullForm]*)

Restore protection of system symbols

Protect[Evaluate[protected]]

End the private context

End[

Epilog

Protect exported symbol

Protect[Evaluate[$Context <> "*"]]

End the package context

EndPackage[

RegularHarmonics is a Mathematica 4.2 package for making computations with Fueter-regular quaternionic functions and harmonic functions of two complex variables. It is based on the results obtained in [S], [P1], and [P2].

Additional information are available on the world wide web at the page [http://www.science.unitn.it/~perotti/regular_harmonics.htm](http://www.science.unitn.it/~perotti/regular_harmonics.htm)

Please send comments and bug reports to: perotti@science.unitn.it.

**Loading the package**

To use the RegularHarmonics package, you have to load it with the command << (or equivalently with *Get*) followed by the name of the .m file. You can use the menu command Input/Get File Path to search for and paste the full pathname of the file RegularHarmonics.m.

```mathematica
<< "C:\...\RegularHarmonics.m"
```

RegularHarmonics by A.Perotti, Version 1.2, April 2004

This package implements computations with Fueter-regular quaternionic polynomials and harmonic functions of two complex variables.

Additional information are available on the world wide web at the page [http://www.science.unitn.it/~perotti/RegularHarmonics.htm](http://www.science.unitn.it/~perotti/RegularHarmonics.htm)

Send comments and bug reports to: perotti@science.unitn.it

**Default variables**

The symbol \( z \) denotes the default indexed complex variable in \( \mathbb{C}^2 \), with two components \( z[1], z[2] \). It is identified with the quaternion \( z_1 + z_2 j \). The complex conjugate \( \text{Conjugate}[z[1]] \) can be input as \( z-\{1\} \) (the conjugation character is obtained with the sequence \( Â– "L \) and is output as \( z_1 \). The same holds for \( \overline{z_2} \).
The symbol $x$ denotes the default indexed real variable with four components $x[0]$, $x[1]$, $x[2]$, $x[3]$. It represents the quaternion $x_0 + i x_1 + j x_2 + k x_3$ and the complex pair $(z_1, z_2) = (x_0 + i x_1, x_2 + i x_3)$.

### Laplacian

Laplacian[$f$, $x$] gives the (ordinary) Laplacian of $f$ with respect to $x$.

\[
\text{Laplacian}[x[1]^2 x[3]^3]\\
2 x_3 (3 x_1^2 + x_2^2)
\]

ComplexLaplacian[$f$, $z$] gives the complex Laplacian of $f$ with respect to $z$. In $\mathbb{C}^2$ it is equal to $1/4$ of the real Laplacian of $f$.

\[
2 z_1
\]

### Field conversions

The following functions perform two-ways conversions between real, complex and quaternionic fields.

RtoC[$(g_1, g_2)$, $x$, $z$] converts the real pair $(g_1, g_2)$ as a function of $x$ to the complex expression $g_1 + i g_2$ as a function of $z$.

\[
x = \text{RtoC}[(x[0], x[1]), x[2]]\\
\frac{1}{4} i z_1^2 + \frac{i z_1}{2} - \frac{i z_2}{4} + \frac{i z_2}{2}
\]

Variables different from the defaults can be given explicitly.

\[
\text{RtoC}[(y[0], y[1]), y, w]\\
w_1
\]
CtoR[\(f, z\)] converts a complex expression \(f\) as a function of \(z\) to the form \{real part, imaginary part\} as a function of \(x\).

\[
\text{CtoR}[\text{cx}]
\]
\[
\{x_0, x_1, x_2\}
\]

CtoH[\(\{f_1, f_2\}, z, x\)] converts the pair \((f_1, f_2)\) as a complex function of \(z\) to the 4–tuple of the real components of the quaternionic expression \(f_1 + f_2 j\).

\[
\text{quat} = \text{CtoH}[\{z[1], z[2], z[-2]\}]
\]
\[
\{x_0, x_1, x_2^3 + x_3^3, 0\}
\]

HtoC[\(\{g_0, g_1, g_2, g_3\}, x, z\)] converts the 4–tuple \((g_0, g_1, g_2, g_3)\) of the real components of a quaternion as a function of \(x\) to the complex pair \((g_0 + i g_1, g_2 + i g_3)\) as a function of \(z\).

\[
\text{HtoC}[\text{quat}]
\]
\[
\{z_1, \overline{z_2} z_2\}
\]

**Cauchy-Riemann-Fueter equations for regular and \(\psi\)-regular functions and related boundary operators**

We refer to [P1] and [P2] for the relevant definitions concerning regular, \(\psi\)-regular quaternionic functions, the Cauchy-Riemann-Fueter equations and the boundary differential conditions characterizing regular functions on a domain in \(\mathbb{C}^2\) among harmonic functions.

\[
\text{CRF}[\{f_1, f_2\}, z] \text{ computes the (left) Cauchy – Riemann – Fueter equations of } f = f_1 + f_2 j \text{ i.e. the pair } \left\{ \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}, \frac{\partial f_1}{\partial z_2} + \frac{\partial f_2}{\partial z_1} \right\}.
\]

\[
\text{CRF}[\{z[1], z[2], z[-2], z[1] + z[2], z[-2]\}]
\]
\[
\{-z_2, \overline{z_2} z_1\}
\]
\(\Psi^{CRF}[\{f_1, f_2\}, z]\) computes the (left) Cauchy–Riemann–Fueter equations for (left) \(\psi\)–regular functions i.e. the pair \(\{\frac{\partial f_1}{\partial \overline{z}_1} - \frac{\partial \overline{f}_2}{\partial z_2}, \frac{\partial f_1}{\partial z_2} + \frac{\partial \overline{f}_2}{\partial \overline{z}_1}\}\).

Note that holomorphic maps of two complex variables define a \(\psi\)–regular function.

\[
\]

\(\{-z_2^2, z_1 z_2\}\)

\[
\]

\(\{0, 0\}\)

The following five differential operators will be used to give boundary differential conditions characterizing regular and \(\psi\)-regular functions on the unit ball in \(\mathbb{C}^2\) among harmonic functions. Cf. [P1] for details.

\(D^{barN}[f, z]\) gives the normal part \(\overline{\partial_n} f = \overline{z}_1 \frac{\partial f}{\partial \overline{z}_1} + \overline{z}_2 \frac{\partial f}{\partial \overline{z}_2}\) of \(\partial f\) with respect to the unit sphere \(S\).

\[
\]

\(\overline{z}_2 z_1 z_2\)

\(L[f, z]\) applies the Cauchy–Riemann tangential (with respect to the unit sphere \(S\)) operator \(z_2 \frac{\partial}{\partial \overline{z}_1} - z_1 \frac{\partial}{\partial z_2}\) to the complex function \(f\).

\[
\]

\(-z_2^2 z_2\)

\(L^{bar}[f, z]\) applies the conjugate Cauchy–Riemann tangential (with respect to the unit sphere \(S\)) operator \(z\overline{z}_2 \frac{\partial}{\partial \overline{z}_1} - \overline{z}_1 \frac{\partial}{\partial z_2}\) to the complex function \(f\).

\[
\]

\(-z_1^2 z_2^2 z_1 + z_2^3 z_2\)
NFueter[f, z] applies the differential operator $N = \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$ to f.

```
```

\[ z_2 z_1 z_2 \]

TFueter[f, z] applies the tangential operator $T = \frac{\partial}{\partial z_2} - z_1 \frac{\partial}{\partial z_1}$ to f.

```
```

\[-z_2^2 z_1\]

RegularQ[{f_1, f_2}, z] tests for (left) Fueter – regularity of $f = f_1 + f_2 j$ on the unit ball B. Here $f$ is a function of $z$, $\bar{z}$.

```
RegularQ[{Sin[z[1]], z[1]^2 z[2]}]
```

False

PsiRegularQ[{f_1, f_2}, z] tests for (left) $\psi$ – regularity of $f = f_1 + f_2 j$ on the unit ball B. Here $f$ is a function of $z$, $\bar{z}$.

```
PsiRegularQ[{Sin[z[1]], z[1]^2 z[2]}]
```

True

---

**Gauss formulas for harmonic extension and harmonic representation of polynomials**

GaussForm[f, z] gives the harmonic representation of the restriction of the polynomial $f(z, \bar{z})$ to the unit sphere S. The output is a list of pairs \( \{h_k, 2k\} \), with $h_k$ harmonic and such that the sum $\sum_k h_k |z|^{2k}$ is equal to $f$ on S. The applied formula can be found for example in the book *Introduction to the theory of cubature formulas* by S.L. Sobolev.
The harmonic extension on the unit sphere $S$.

$$H_{1}z - Lz = z^{2}z - Lz$$

and

$$\gamma$$

gives a regular or $\gamma$-regular extension to the unit sphere $S$. It is based on the harmonic representation of $f$ (see above the function $\text{GaussForm}[f, z]$).


$$\frac{2}{3}z_{1}^{2} + \frac{1}{3}\overline{z_{1}}z_{1}^{2} - \frac{2}{3}\overline{z_{1}}z_{1}z_{2} - \frac{3}{2}\overline{z_{1}}z_{1}z_{2}^{2} + \frac{3}{2}\overline{z_{1}}z_{2} + \frac{1}{2}\overline{z_{2}}z_{2}^{2}$$

The restriction of the polynomial to the unit sphere $S$ can be computed by means of the function $\text{OnS}$. 

$$\text{OnS}[ff - f]$$

$$0$$

$\text{ExteriorGaussExtension}[f, z]$ gives the harmonic extension on the complement of the unit ball of the restriction of the polynomial $f[z, \bar{z}]$ to the unit sphere $S$.

$$\text{ToComplexNorm}[\text{ExteriorGaussExtension}[f]]$$

$$\frac{|z|^{2}(2|z|^{2}z_{1} + 4|z|^{4}z_{1}z_{2} + 9z_{2}^{2} - 9|z|^{2}z_{2}^{2}) - 6z\overline{z}(|z|^{2}z_{1}z_{2} + 2z_{2})}{6|z|^{10}}$$

**Regular and $\psi$-regular extensions of polynomials on the unit ball**

The following functions use the results given in [P1] and [P2] in order to obtain regular and $\psi$-regular extension of polynomials.

$$\text{RegularExtension}[[h_{1}, h_{2}], z]$$

and

$$\text{PsiRegularExtension}[[h_{1}, h_{2}], z]$$
give, if they exist, the (left) regular and the $\psi$-regular extension $f = f_{1} + f_{2}j$ of the restriction of $h = h_{1} + h_{2}j$ to the unit sphere.

$$\text{RegularExtension}[h_{1}, z]$$

and

$$\text{PsiRegularExtension}[h_{1}, z]$$
gives a regular or $\psi$-regular polynomial $f = f_{1} + f_{2}j$ such that $f_{1} = h_{1}$ on the unit sphere.

Here $f_{1}$, $f_{2}$ and $h_{1}$ must be polynomial functions of $z$, $\bar{z}$. The output is the pair of complex components $\{f_{1}, f_{2}\}$ of $f$.
\[ \text{RegularExtension}[[z[1]^4 z-[1]^3, z[1]]] \]

No regular extension

\( ff = \text{RegularExtension}[z[1]^4 z-[1]^3] \)

\[ \left\{ \frac{2 z_1}{5} + \frac{2}{5} z_1^2 z_2^2 + \frac{6}{35} z_1^3 z_2, \frac{1}{35} z_1^3 z_1^3 - \frac{4}{5} z_1^2 z_1 z_2 - \frac{36}{35} z_1^2 z_2^3, \frac{1}{2} z_1^2 z_2 + \frac{12}{35} z_1^2 z_1 z_2 + \frac{18}{35} z_1 z_2^2 + \frac{18}{35} z_1 z_2^3 z_2^2 - \frac{4}{35} z_1 z_1^2 z_1^2 z_2^2, \frac{2}{5} z_1^2 z_2 + \frac{12}{35} z_1^2 z_1 z_2 + \frac{3}{35} z_1 z_2^2 - \frac{18}{35} z_1 z_2^2 z_2^2 - \frac{12}{35} z_1^3 z_1^2 z_2^2 + \frac{6}{35} z_1^3 z_2^2 z_2^2 \right\} \]

\( \text{OnS}[ff] \)

\[ \left\{ -z_1 (-1 + z_2^2)^3, \frac{1}{35} z_1^2 z_2 (29 - 48 z_2^2 + 21 z_2^2 z_2^2) \right\} \]

\( \text{PsiRegularExtension}[[z[1]^4 z-[1]^3, z[1]]] \)

No \( \psi \)-regular extension

\( g = \text{PsiRegularExtension}[z-[1] z[1]^2] \)

\[ \left\{ \frac{2 z_1}{3} + \frac{1}{3} z_1^2 z_2, \frac{2}{3} z_1 z_2, \frac{1}{3} z_1^2 z_2 \right\} \]

The function \( \text{CtoH} \) can be applied to get the four real component of the quaternionic function whose complex components have been computed above.

\( \text{CtoH}[g] \)

\[ \left\{ \frac{2 x_0}{3} + \frac{x_0^2}{3}, \frac{1}{3} x_0 x_1 - \frac{2}{3} x_2, \frac{2}{3} x_0 x_2, \frac{2}{3} x_1 + \frac{1}{3} x_0 x_2, \frac{2}{3} x_0 x_1 + \frac{x_1^2}{3}, -\frac{2}{3} x_1 x_2 - \frac{2}{3} x_1 x_3, \frac{1}{3} x_0 x_2 - \frac{1}{3} x_0 x_1 x_3 - \frac{2}{3} x_0 x_1 x_3 - \frac{1}{3} x_0 x_3 - \frac{1}{3} x_0 x_3 \right\} \]
Sphere and ball products and norms

SphereIntegral[f, z] and BallIntegral[f, z] give the normalized integral over the unit sphere S (resp. the unit ball B) of the polynomial f[z, \bar{z}]. The volume of S and B are normalized to 1.

\[
\text{SphereIntegral}[z[1]^3 z-[1]^3 z[2] z-[2]]
\]
\[
\frac{1}{20}
\]

\[
\text{BallIntegral}[z[1]^3 z-[1]^3 z[2] z-[2]]
\]
\[
\frac{1}{60}
\]

SphereProduct[f, g, z] and BallProduct[f, g, z] give the normalized L^2 product over the unit sphere S (resp. the unit ball B) of the complex polynomials f[z, \bar{z}] and g[z, \bar{z}].

\[
\text{SphereProduct}[\{f_1, f_2\}, \{g_1, g_2\}, z]
\]
\[
\text{BallProduct}[\{f_1, f_2\}, \{g_1, g_2\}, z]
\]
give the normalized L^2 product over the unit sphere S (resp. the unit ball B) of the quaternionic polynomials f_1 + f_2 j and g_1 + g_2 j.

SphereNorm[f, z] and BallNorm[f, z] give the normalized L^2 norm of the polynomial f[z, \bar{z}].

\[
\text{SphereNorm}[\{f_1, f_2\}, z]
\]
\[
\text{BallNorm}[\{f_1, f_2\}, z]
\]
gives the normalized L^2 norm of the quaternionic polynomial f_1 + f_2 j.

\[
\text{SphereNorm}[\text{RegularExtension}[z[1]^3 z-[1]]]
\]
\[
\frac{\sqrt{13}}{8}
\]

\[
\text{BallNorm}[\text{RegularExtension}[z[1]^3 z-[1]]]
\]
\[
\frac{\sqrt{13}}{8}
\]
Spherical harmonics bases

\textbf{BasisP}[p, q, z] \text{ gives a basis of the space } \mathcal{H}_p, q \text{ of the complex harmonic homogeneous polynomials of degree } p \text{ in } z_1, z_2 \text{ and } q \text{ in } \overline{z_1}, \overline{z_2}.

It is a \( L^2(S) \) – orthogonal basis introduced by Sudbery (see References).

\textbf{BasisP}[k, z] \text{ gives a basis of the space } \mathcal{H}_k = \bigoplus \mathcal{H}_p, q \text{ of the complex harmonic homogeneous polynomials of degree } k.

\textbf{RealBasisP}[k, z] \text{ gives a real basis of the space } \mathcal{H}_k \text{ of the complex harmonic homogeneous polynomials of degree } k \text{ in } z.

\textbf{b} = \text{RealBasisP}[3]

\begin{align*}
&\{ \overline{z}_1^3 z_1^3, -3 \overline{z}_1^2 z_2^2 z_1^2 + 2 \overline{z}_2^2 z_1 z_2, 3 \overline{z}_1^2 z_2^2 z_1^2 - 6 \overline{z}_1^2 z_2^2 z_1 z_2 + \overline{z}_2^3 z_2^2, \\
&\overline{z}_1^3 z_1^3 - 6 \overline{z}_1^2 z_2 z_1 z_2 + 3 \overline{z}_2 z_1^2 z_2^2, 2 \overline{z}_1^3 z_1 z_2 - 3 \overline{z}_1^2 z_2^2 z_2^2, \overline{z}_1^3 z_2^3 \}
\end{align*}

\textbf{CtoR[b][[1]]}

\begin{align*}
&\{ 2 x_0^3 - 6 x_0 x_1^3, 6 x_0^3 x_2 - 6 x_1^2 x_2 - 12 x_0 x_1 x_3, 6 x_0 x_2^2 - 12 x_1 x_2 x_3 - 6 x_0 x_3^2, \\
&2 x_0^3 - 6 x_2 x_1^3, 2 x_1^3 x_2 - 2 x_2^3 x_1 x_3 + 4 x_0 x_1 x_3, -2 x_0^3 - 2 x_1^2 x_2 + 4 x_0 x_2^2 + 4 x_0 x_3^2, \\
&4 x_0^3 x_2 + 4 x_1^3 x_2 - 2 x_2^3 + 2 x_2 x_3^2 - 2 x_0 x_2 x_3 - 2 x_0 x_3^3, 6 x_0 x_3^3 + 4 x_1 x_3^3 + 4 x_1 x_2 x_3, \\
&12 x_0 x_1 x_2 + 6 x_1 x_2^3 - 6 x_1^3 x_3, 6 x_1 x_3^2 + 12 x_0 x_2 x_3 - 6 x_1 x_3^2, 6 x_2^3 + 2 x_1^2 x_3, \\
&4 x_0 x_1 x_3 + 2 x_1^3 x_2 - 2 x_2 x_3^3 - 2 x_0 x_2 x_3 + 2 x_0 x_3^3, 4 x_2 x_3^3 + 2 x_1^3 x_3, \\
&4 x_0 x_2 x_3 - 2 x_1^3 x_2, -2 x_0 x_1 x_3 + 2 x_0^3 + 4 x_1 x_3^3 + 4 x_1 x_2 x_3, \\
&4 x_0 x_3^3 + 4 x_1 x_3^3 + 2 x_2 x_3^3 - 2 x_1^3, -2 x_1^3 + 2 x_0 x_2 x_3 + 2 x_1 x_3^3 \}
\end{align*}

\textbf{ONBasisP}[p, q, z] \text{ gives a } L^2(S) – \text{ orthonormal basis of the space } \mathcal{H}_p, q \cdot \textbf{ONBasisP}[k, z] \text{ gives a } L^2(S) – \text{ orthonormal basis of the space } \mathcal{H}_k.
BallONBasisP[p, q, z] gives a $L^2(B)$ – orthonormal basis of the space $H_p, q$. BallONBasisP[k, z] gives a $L^2(B)$ – orthonormal basis of the space $H_k$.

ONBasisP[2, 3]

$$\left\{2 \sqrt{15} \bar{z}_1^2 \bar{z}_2^3, 2 \sqrt{3} \left(-3 \bar{z}_1 \bar{z}_2^2 \bar{z}_3^2 + 2 \bar{z}_2^3 \bar{z}_1 \bar{z}_2\right), \sqrt{6} \left(3 \bar{z}_1^2 \bar{z}_2^2 \bar{z}_1^2 - 6 \bar{z}_1 \bar{z}_2^2 \bar{z}_1 \bar{z}_2 + 3 \bar{z}_1 \bar{z}_2^2 \bar{z}_2^2\right), \sqrt{2} \left(2 \bar{z}_1 \bar{z}_2 - 3 \bar{z}_1 \bar{z}_2^2 \bar{z}_2^2\right), 2 \sqrt{15} \bar{z}_2^3 \right\}$$

BallONBasisP[2, 3]

$$\left\{\sqrt{210} \bar{z}_1^3 \bar{z}_2^3, \sqrt{42} \left(-3 \bar{z}_1 \bar{z}_2^2 \bar{z}_3^2 + 2 \bar{z}_2^3 \bar{z}_1 \bar{z}_2\right), \sqrt{21} \left(3 \bar{z}_1^2 \bar{z}_2^2 \bar{z}_1^2 - 6 \bar{z}_1 \bar{z}_2^2 \bar{z}_1 \bar{z}_2 + 3 \bar{z}_1 \bar{z}_2^2 \bar{z}_2^2\right), \sqrt{42} \left(2 \bar{z}_1 \bar{z}_2 - 3 \bar{z}_1 \bar{z}_2^2 \bar{z}_2^2\right), \sqrt{210} \bar{z}_2^3 \right\}$$

Regular and $\psi$-regular spherical harmonics bases

RegularBasis[k, z] gives a basis of the right quaternionic module $U_k$ of the (left) regular homogeneous polynomials of degree $k$ in $z$.

PsiRegularBasis[k, z] gives a basis of the right quaternionic module $U_k^\psi$ of the (left) $\psi$ – regular homogeneous polynomials of degree $k$ in $z$.

The restrictions to $S$ gives a basis of the regular harmonics.

RegularONBasis[k, z] and RegularBallONBasis[k, z]
give orthonormal bases of the right quaternionic module $U_k$.

PsiRegularONBasis[k, z] and PsiRegularBallONBasis[k, z]
give orthonormal bases of the right quaternionic module $U_k^\psi$. 
RegularBasis[4]

\[
\{(z_1^4, 0), (4 z_1^2 z_2^1, 0), (6 z_2^0 z_1^2, 0), (4 z_3^3 z_1, 0), \\
(z_2^4, 0), (z_1^3 z_2, -\frac{z_3^4}{4}), (-z_1^3 z_2^1 + 3 z_2^0 z_1^2, -z_1^3 z_2), \\
(3 z_1^3 z_2^1 - 3 z_2^0 z_1 z_2, \frac{3}{2} z_1^2 z_2^2), (3 z_1^3 z_2 - z_2^0 z_1 z_2, z_1^2 z_2), (z_1^2 z_2^3, \frac{z_4^4}{4}), \\
(z_2^4 z_2^2, -\frac{2}{3} z_1^0 z_2^2), (-2 z_1^0 z_2 z_2 + 2 z_1 z_2^1 z_2, \frac{2}{3} z_1^0 z_1 - 2 z_1^2 z_2^2), \\
(z_1^2 z_2^2 - 4 z_1 z_2^1 z_2 + z_2^0 z_2^2, 2 z_1^2 z_1 z_2 - 2 z_1^0 z_2^2 z_2), \\
(2 z_1^2 z_2 z_2 - 2 z_1^0 z_2^2 z_2, 2 z_1 z_2^1 z_2^2 - \frac{2}{3} z_1^0 z_2^3), (z_1^2 z_2^2, \frac{2}{3} z_1^0 z_2^3)\}
\]

\[\text{PsiRegularBasis}[3]\]

\[
\{(z_1^4, 0), (3 z_1^2 z_2^2, 0), (3 z_1 z_2^0, 0), (z_2^2, 0), (z_2^0 z_2^2, -\frac{z_3^3}{3}), \\
(-z_1^3 z_2 + 2 z_1 z_2^2, -z_2^0 z_2^2), (2 z_1 z_2 - z_1 z_2^2 z_2 + z_2^0 z_1^2), (z_1 z_2^2, \frac{z_3^3}{3}), \\
z_2^2 z_1, -z_1^2 z_2), (-2 z_1 z_2^2 z_1 + z_2^2 z_2, z_1^2 z_1 - 2 z_1 z_2^2 z_2)\}
\]

\[\text{PsiRegularONBasis}[4]\]

\[
\{(\sqrt{5} z_1^4, 0), (2 \sqrt{5} z_1^2 z_2^2, 0), (\sqrt{30} z_1^2 z_2^2, 0), (2 \sqrt{5} z_1 z_2^1, 0), \\
(\sqrt{5} z_2^0, 0), (4 z_2^0 z_1^2, -z_2^4), (-2 z_1 z_2 + 6 z_2^0 z_1^2 z_2, -2 z_1^3 z_1 z_2^1), \\
(2 \sqrt{6} z_1^2 z_2^2 - 2 \sqrt{6} z_1 z_2^1 z_2^1, 6 z_1 z_2^1 z_2^2 + 6 z_1 z_2^3), (6 z_1 z_2^1 z_2^2, 2 z_1 z_2^3), \\
(4 z_1 z_2^2, z_2^4), (3 \sqrt{2} z_2^4 z_2^1, 3), -2 \sqrt{2} z_1^3 z_2^1), \\
(-3 \sqrt{2} z_1 z_2^2 z_1^3 + 3 \sqrt{2} z_1 z_2^2 z_2^1, \sqrt{2} z_1^3 z_1 z_2 - 3 \sqrt{2} z_1^3 z_2^2 z_2), \\
(3 \sqrt{3} z_1^2 z_1^2 z_2^2 - 4 \sqrt{3} z_1 z_2^1 z_1^2 z_2^1 + \sqrt{3} z_1 z_2^0 z_2^2, 2 \sqrt{3} z_1 z_2^1 z_1^2 z_2, 2 \sqrt{3} z_1 z_2^0 z_2^2 z_2), \\
(3 \sqrt{2} z_1^2 z_1 z_2 - 3 \sqrt{2} z_1 z_2^2 z_1^3, 3 \sqrt{2} z_1 z_2^2 z_2 z_1 - \sqrt{2} z_1 z_2^3 z_2), \\
(3 \sqrt{2} z_1 z_2^2 z_2, 2 \sqrt{2} z_2^3 z_1)\}
\]

Help

To get the usage message of a package function, evaluate the input ?FunctionName.

\[
n:=\text{Names}[^\"RegularHarmonics\"^];\text{ToExpression}[\text{Table}[\text{StringJoin}[^{\"?\", \text{ToString}[\text{Part}[n,i]]}],\{i,\text{Length}[n]\}]];\]
BallIntegral[f,z] gives the normalized integral over the unit ball B of the polynomial f[z,z].

The volume of B is assumed to be 1

BallNorm[f,z] gives the normalized $L^2$ norm over the unit ball B of the polynomial f[z,z].

BallNorm[{f1,f2},z] gives the normalized $L^2$ norm over the unit ball B of the quaternionic polynomial $f_1+f_2j$.

The volume of B is assumed to be 1

BallONBasisP[p,q,z] gives a $L^2(B)$-orthonormal basis of the space $\mathcal{H}_{p,q}$ of the complex harmonic homogeneous polynomials of degree $p$ in $z_1, z_2$ and $q$ in $\mathbb{H}, \mathbb{H}^2$.

It is obtained from a basis introduced by Sudbery (see References).

BallONBasisP[k,z] gives a $L^2(B)$-orthonormal basis of the space $\mathcal{H}_k = \bigoplus_{p,q} \mathcal{H}_{p,q}$ of the complex harmonic homogeneous polynomials of degree $k$.

BallProduct[f,q,z] gives the normalized $L^2$ product over the unit ball $B$ of the complex polynomials $f[z,z]$ and $q[z,z]$.

BallProduct[{f1,f2},{q1,q2},z] gives the normalized $L^2$ product over the unit ball $B$ of the quaternionic polynomials $f_1+f_2j$ and $q_1+q_2j$.

The volume of B is assumed to be 1

BasisP[p,q,z] gives a basis of the space $\mathcal{H}_{p,q}$ of the complex harmonic homogeneous polynomials of degree $p$ in $z_1, z_2$ and $q$ in $\mathbb{H}, \mathbb{H}^2$.

It is a $L^2(S)$-orthogonal basis introduced by Sudbery (see References).

BasisP[k,z] gives a basis of the space $\mathcal{H}_k = \bigoplus_{p,q} \mathcal{H}_{p,q}$ of the complex harmonic homogeneous polynomials of degree $k$.

ComplexLaplacian[f,z] gives the complex Laplacian of $f$ with respect to $z$. In $\mathbb{C}^2$ it is equal to 1/4 of the real Laplacian of $f$

CRF[{f1,f2},z] computes the (left) Cauchy-Riemann-Fueter equations of $f=f_1+f_2j$ i.e. the pair \( \left\{ \frac{\partial f_1}{\partial z_1}, \frac{\partial f_1}{\partial z_2}, \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \right\} \)

CtoH[{f1,f2},z,x] converts the pair \{f1,f2\} as a complex function of $z_1=x_0+ix_1$ and $z_2=x_2+ix_3$ to the 4-tuple of the real components of the quaternion $f_1+f_2j$

CtoR[f,z,x] converts a complex expression $f[z,z]$ as a function of $z_1=x_0+ix_1$ and $z_2=x_2+ix_3$ to the form (real part, imaginary part) in terms of $x_0, x_1, x_2, x_3$

DbarN[f,z] gives the normal part $\bar{\partial}_n f = \bar{x}_1 \frac{\partial f}{\partial \bar{x}_1} + \bar{x}_2 \frac{\partial f}{\partial \bar{x}_2}$ of $\partial f$ with respect to the unit sphere $S$

ExteriorGaussExtension[f,z] gives the harmonic extension on the complement of the unit ball of the restriction of the polynomial $f[z,z]$ to the unit sphere $S$

GaussExtension[f,z] gives the (polynomial) harmonic extension of the restriction of the polynomial $f[z,z]$ to the unit sphere $S$
GaussForm[f, z] gives the harmonic representation of the restriction of the polynomial f[z, z] to the unit sphere S. The output is a list of pairs \((h_k, 2k)\), with \(h_k\) harmonic and such that the sum \(\sum h_k |z|^{2k}\) is equal to \(f\) on \(S\).

HtoC\([\{g_0, g_1, g_2, g_3\}, x, z]\) converts the 4-tuple \([g_0, g_1, g_2, g_3]\) as a function of \(x\) to the complex pair \([g_0+ig_1, g_2+ig_3]\) as a function of \(z, \bar{z}\).

KelvinTransform[f, z] gives the Kelvin Transform of \(f\) to \(S\) in \(C^2\).

L[f, z] applies the Cauchy-Riemann tangential operator \(z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2}\) to \(f\).

Lbar[f, z] applies the conjugate Cauchy-Riemann tangential operator \(\overline{z_2} \frac{\partial}{\partial z_1} - \overline{z_1} \frac{\partial}{\partial z_2}\) to \(f\).

LeadingTerm[f, z] gives the leading term of the polynomial \(f[z, \bar{z}]\) with respect to the graded lexicographic order with \(z_1 > z_2 > \overline{z_1} > \overline{z_2}\).

NFueter[f, z] applies the differential operator \(N = \overline{z_2} \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}\) to \(f\).

ONBasisP[p, q, z] gives a \(L^2(S)\)-orthonormal basis of the space \(\mathcal{H}_{p,q}\) of the complex harmonic homogeneous polynomials of degree \(p\) in \(z_1, z_2\) and \(q\) in \(\overline{z_1}, \overline{z_2}\).

It is obtained from a basis introduced by Sudbery (see References).

ONBasisP[k, z] gives a \(L^2(S)\)-orthonormal basis of the space \(\mathcal{H}_k = \bigoplus \mathcal{H}_{p,q}\) of the complex harmonic homogeneous polynomials of degree \(k\).

OnS[f, z] computes the restriction of \(f[z, \bar{z}]\) to the unit sphere \(S\) in \(C^2\).

PsiCRF\([\{f_1, f_2\}, z]\) computes the (left) Cauchy-Riemann-Fueter equations for (left) \(\psi\)-regular functions i.e. the pair \(\left\{\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}, \frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}}\right\}\).

PsiRegularBallONBasis[k, z] gives a \(L^2(B)\)-orthonormal basis of the right quaternionic module \(U_k\) of the (left) \(\psi\)-regular homogeneous polynomials of degree \(k\) in \(z\).

PsiRegularBasis[k, z] gives a basis of the right quaternionic module \(U_k\) of the (left) \(\psi\)-regular homogeneous polynomials of degree \(k\) in \(z\).

The restrictions to \(S\) gives a basis of the regular harmonics.

PsiRegularExtension\([\{f_1, f_2\}, z]\) gives, if it exists, the (left) \(\psi\)-regular extension of the restriction of \(f=f_1+f_2j\) to the unit sphere.

PsiRegularExtension\([h_1, z]\) gives a \(\psi\)-regular polynomial \(f=f_1+f_2j\) such that \(f_1=h_1\) on the unit sphere.

Here \(f_1, f_2\) and \(h_1\) must be polynomial functions of \(z, \bar{z}\).

PsiRegularExtensionQ\([\{f_1, f_2\}, z]\) tests for (left) \(\psi\)-regularity of the harmonic extension of the restriction of \(f=f_1+f_2j\) to the unit sphere. Here \(f_1\) and \(f_2\) must be polynomial functions of \(z, \bar{z}\).
PsiRegularONBasis[k,z] gives a $L^2(S)$-orthonormal basis of the right quaternionic module $U_k^\circ$ of the (left) $\psi$-regular homogeneous polynomials of degree $k$ in $z$.

PsiRegularQ[{f_1,f_2},z] tests for (left)$\psi$-regularity of $f=f_1+f_2 j$. Here $f$ is a function of $z,z^\ast$.

RealBallONBasisP[k,z] gives a $L^2(B)$-orthonormal real basis of the space $H_k$ of the complex harmonic homogeneous polynomials of degree $k$ in $z$.

It is obtained from a complex basis introduced by Sudbery (see References).

RealBasisP[k,z] gives a real basis of the space $\mathcal{H}_k$ of the complex harmonic homogeneous polynomials of degree $k$ in $z$.

It is a $L^2$-orthogonal basis obtained from a complex basis introduced by Sudbery (see References).

RealONBasisP[k,z] gives a $L^2(S)$-orthonormal real basis of the space $\mathcal{H}_k$ of the complex harmonic homogeneous polynomials of degree $k$ in $z$.

It is obtained from a complex basis introduced by Sudbery (see References).

RegularBallONBasis[k,z] gives a $L^2(B)$-orthonormal basis of the right quaternionic module $U_k^\circ$ of the (left) regular homogeneous polynomials of degree $k$ in $z$.

RegularBasis[k,z] gives a basis of the right quaternionic module $U_k^\circ$ of the (left) regular homogeneous polynomials of degree $k$ in $z$.

RegularExtension[{f_1,f_2},z] gives, if it exists, the (left) regular extension of the restriction of $f=f_1+f_2 j$ to the unit sphere.

RegularExtension[h_1,z] gives a regular polynomial $f=f_1+f_2 j$ such that $f_1=h_1$ on the unit sphere.

Here $f_1,f_2$ and $h_1$ must be polynomial functions of $z,z^\ast$.

RegularExtensionQ[{f_1,f_2},z] tests for (left)Fueter-regularity of the harmonic extension of the restriction of $f=f_1+f_2 j$ to the unit sphere. Here $f_1$ and $f_2$ must be polynomial functions of $z,z^\ast$.

RegularHarmonics.m is a package that implements computations with (left)regular quaternionic polynomials and harmonic functions of two complex variables.

RegularONBasis[k,z] gives a $L^2(S)$-orthonormal basis of the right quaternionic module $U_k^\circ$ of the (left) regular homogeneous polynomials of degree $k$ in $z$.

RegularQ[{f_1,f_2},z] tests for (left)Fueter-regularity of $f=f_1+f_2 j$. Here $f$ is a function of $z,z^\ast$.

RtoC[{g_1,g_2},x,z] converts the real pair $\{g_1,g_2\}$ as a function of $x_0,x_1,x_2,x_3$ to the complex expression $g_1+ig_2$ as a function of $z_1=x_0+ix_1$ and $z_2=x_2+ix_3$.

SphereIntegral[f,z] gives the normalized integral over the unit sphere $S$ of the polynomial $f[z,z^\ast]$.

The volume of $S$ is assumed to be 1.
SphereNorm[f,z] gives the normalized $L^2$ norm over the unit sphere $S$ of the polynomial $f[z,z]$.
SphereNorm[{f₁,f₂},z] gives the normalized $L^2$ norm over the unit sphere $S$ of the quaternionic polynomial $f₁+f₂j$.
The volume of $S$ is assumed to be 1

SphereProduct[f,g,z] gives the normalized $L^2$ product over the unit sphere $S$ of the complex polynomials $f[z,z]$ and $g[z,z]$.
SphereProduct[{f₁,f₂},{g₁,g₂},z] gives the normalized $L^2$ product over the unit sphere $S$ of the quaternionic polynomials $f₁+f₂j$ and $g₁+g₂j$.
The volume of $S$ is assumed to be 1

TFueter[f,z] applies the tangential operator $T=\overline{z²} \frac{\partial}{\partial \overline{z₁}} - z₁ \frac{\partial}{\partial z₂}$ to $f$

ToComplexNorm[f,z] converts the expression $f$ in terms of the norms of $z₁,z[1],z[2]$

If Tonorm has value True, subsequent calls to many functions of the package express results in terms of the norms of $z₁,z[1],z[2]$

ToRealNorm[f,x] converts the expression $f$ in terms of the norm of $x$

TotalDegree[f,z] gives the total degree of a polynomial $f$ in $z₁,z₂$

$x$ is the default indexed real variable with four components $x[0],x[1],x[2],x[3]$; it represents the quaternion $x₀+ix₁+jx₂+kx₃$

$z$ is the default indexed complex variable in $C²$ with two components $z₁,z[2]$; it represents the quaternion $z₁+z[2]j$.
Conjugate[$z[1]$] is input as $z-[1]$ and output as $\overline{z[1]}$. The same for $\overline{z[2]}$.

References


- http://www.science.unitn.it/~perotti/regular harmonics.htm